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# Extended fermion representation of multi-charge 1/2-BPS operators in AdS/CFT — towards field theory of D-branes —

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Abstract: We extend the fermion representation of single-charge 1/2-BPS operators in the four-dimensional $\mathcal{N}=4$ super Yang-Mills theory to general (multi-charge) 1/2-BPS operators such that all six directions of scalar fields play roles on an equal footing. This enables us to construct a field-theoretic representation for a second-quantized system of spherical D3-branes in the 1/2-BPS sector. The Fock space of D3-branes is characterized by a novel exclusion principle (called 'D'exclusion' principle), and also by a nonlocality which is consistent with the spacetime uncertainty relation. The Dexclusion principle is realized by composites of two operators, obeying the usual canonical anticommutation relation and the Cuntz algebra, respectively. The nonlocality appears as a consequence of a superselection rule associated with a symmetry which is related to the scale invariance of the super Yang-Mills theory. The entropy of the so-called superstars, with multiple charges, which have been proposed to be geometries corresponding to the condensation of giant gravitons is discussed from our viewpoint and is argued to be consistent with the Dexclusion principle. Our construction may be regarded as a first step towards a possible new framework of general D-brane field theory.

Keywords: Black Holes in String Theory, D-branes, Brane Dynamics in Gauge Theories, AdS-CFT Correspondence.



 

## Contents



## 1. Introduction

Ever since D-branes were recognized as the crucial elements of string/M theory, one of difficult problems has been the precise description of dynamical creation and annihilation of (anti-) D-branes. In order to make real progress towards a satisfactory nonperturbative formulation of string/M theory, it is desirable to develop a new framework in which such dynamical processes can be appropriately treated.

So far, we have two main methods for discussing the dynamics of D-branes. One is from the viewpoint of open-string theory in which the boundary conditions at the open-string ends are explicitly taken into account. Various modes of open strings are then interpreted as describing all the possible dynamical degrees of freedom of D-branes. In particular, their transverse positions correspond to the lowest massless modes. As such, the open-string field theory is a first-quantized 'configuration-space' formulation of D-branes, even though it is second-quantized as the quantum theory of strings. The (effective) Yang-Mills theories of D-branes, or matrix models, are in the same vein as open-string field theories.

In closed-string (field) theories, on the other hand, D-branes are interpreted as a kind of soliton-like excitations. A related viewpoint on D-branes has been provided from the K-theory classification [1] of D-branes. In this case, we start from an appropriate large $N$ limit of D9-brane systems, since we can embed in them an arbitrary number of stable or unstable (anti) D-branes as solitons or "lump" solutions. Qualitatively similar structure has also been appearing in the so-called vacuum string-field theory [2] of open strings, in which there is no propagating degree of freedom for open strings, but (unstable) D-branes again emerge as soliton-like solutions.

In all these cases of the second category, the situation is analogous to discussing the creation and annihilation of kinks by using sine-Gordon field theory in the case of ordinary field theory in two dimensions. For kinks, it is well known that the system is equivalently described by the massive Thirring model in which kinks are now regarded as elementary excitations corresponding to the Dirac field. The latter is more natural and convenient when we have to take into account a large fluctuation with respect to pair creation and annihilation of kinks. Conversely, the sine-Gordon field theory is obtained by the bosonization of the massive Thirring model. In the case of D-branes, using this analogy, we do not have the formulation corresponding to the massive Thirring model, namely, the field theory of D-branes, in which D-branes are treated as elementary excitations.

This is one of the basic motivations of the present work: The question we propose to pursue is whether and how it is possible to second-quantize the open-string or Yang-Mills description mentioned above. The second quantization in ordinary particle quantum mechanics generalizes the Hilbert space of states with fixed number of particles by introducing the Fock space representation in which all Hilbert spaces with different particle number are treated in a unified way. The single-particle wave functions are elevated to quantum field operators acting upon the Fock space as agents creating or annihilating particles. Similarly, we may introduce the Fock-type generalized Hilbert space and quantum field operators creating and annihilating D-branes, by which we treat the whole space of super Yang-Mills theories with different $N$ in a unified operator formalism.

At first sight, such an attempt might look rather bizarre in view of expected weirdness of field theories for extended objects, except when we restrict ourselves to certain very limited topological aspects of D-branes. However, from the viewpoint of $\mathrm{AdS}_5\times \mathrm{S}^5/\mathrm{MSYM}_4$ correspondence, feasibility of such a dynamical structure, at least to some extent, is suggested by a recent development on the description of 1/2-BPS operators on both sides of bulk and boundary theories.

It has been argued that a special class of 1/2-BPS operators in $\mathrm{U}(N)$ super Yang-Mills theory, which are characterized by the condition $\Delta$(conformal dimension) $= J$ with $J$ being the angular momentum along a specially chosen U(1) direction in $\mathrm{S}^5$, are described by a matrix model [4, 5] with a single (complex) $N\times N$ matrix field $Z(t)$ in one dimension, which is reminiscent of the old c=1 matrix model. This matrix model is equivalently described by $N$ free fermions. The second quantization of these fermions may bring us something close to the D-brane field, since $N$ is nothing but the total D3-brane charge.

This view has been considerably boosted by a more recent remarkable work by Lin, Lunin, and Maldacena [6]. They showed that the supergravity solultions satisfying the same symmetry conditions as these 1/2-BPS operators are completely classified, under a certain smoothness requirement, by a definite boundary condition which is formulated on a special two-dimensional plane embedded in the bulk. The boundary condition for each supergravity solution without singularity specifies a configuration which is interpreted as the classical phase space configuration corresponding to a quantum state of $N$ fermions of the matrix model. Quite remarkably, there is a one-to-one holographic correspondence between microstates defined at the AdS boundary using fermions and the smooth classical solutions in bulk supergravity.

Actually, as we discuss in the next section 2, the identification of these fermions precisely as D3-branes on the gauge-theory side suffers from a puzzle which prevents us from making such a simple-minded conclusion. In the present paper, we propose an entirely new viewpoint for the fermion picture, on deriving it for *generic* 1/2-BPS operators by treating all six directions of scalar fields $\phi_i$ of MSYM$_4$ theory on an equal footing. This naturally resolves the puzzle and makes us possible to start for a quantum-field theory of D3-branes. The construction of field theoretical representation of spherical D3-branes in the 1/2-BPS sector given below is hopefully a small but a first step towards our goal of establishing a possible new framework for the dynamics of D-branes.

The paper is organized as follows. In the next section 2, we give a critical review on the well known free-fermion representation of 1/2-BPS operators from the viewpoint of possible D-brane field theory. In section 3, we first discuss the simple factorization property of 2-point functions of generic 1/2 BPS operators, which is acturally also valid for general 3-point functions and the case of higher-point extremal correlators. It is shown that the emergence of the fermion picture is essentially due to the separation of the SO(6) degrees of freedom from the real dynamical structure which contains the dependence of the dynamics on the number of D-branes. We show that a generalized version of Pauli's exclusion principle must be operative, and give an operator realization of this D-brane exclusion principle ('Dexclusion principle' in short), by introducing composite fermion operators using the Cuntz algebra as well as the ordinary canonical fermion algebras. In section 4, we construct the quantum fields of spherical D3-branes whose base spacetime is effectively seven dimensional, and discuss the properties of their bilinears as creation and annihilation operators of giant (and ordinary) gravitons. We show that they reproduce correctly the two-point functions and general extremal correlators. We then discuss the nature of nonlocality which is consistent with the spacetime uncertainty relation. In section 5, we first briefly consider the meaning of the Dexclusion principle from the viewpoint of bulk supergravity. We then discuss the connection of our formulation of mutiple-charged 1/2 BPS operators to the so-called superstar solutions with mutiple charges on the supergravity side. We argue that the entropy of the superstars satisfies an inequaltity, which is consitent with the Dexclusion principle. The final section 6 is devoted to a summary and concluding remarks on future problems. In the appendix, we give a derivation of the fermion picture in the usual approach, for the purpose of setting up our notations and for convenience of the reader in comparing our methods with other works.

## 2. A critical review: Free fermion as the quantum field of D3-branes?

A low-energy effective description of (stable) multiple D-branes is given by matrix fields on their world volume, $X^{\mu}_{ab}$ and their superpartners, where $a, b, \ldots$ run from 1 to $N$ with $N$ being the number of branes or RR-charge and $\mu$ represents the spacetime directions. In particular, the transverse positions of D-branes are described by the diagonal elements of scalar directions ($\mu = i$, $i = 1, 2, \ldots, 10 - p - 1$), whereas the off-diagonal elements represent the lowest modes of open strings connecting different D-branes. The theory has the Chan-Paton gauge symmetry $X^{\mu} \to UX^{\mu}U^{-1}$, $U \in \mathrm{U}(N)$. The gauge symmetry is regarded as the generalization of permutation symmetry of multi-particle states in the case of ordinary particles, since the gauge transformation is reduced to permutation of the diagonal elements when the matrix fields are diagonalized.

The most crucial property of the ordinary second quantization is that the quantum statistical property of multi-particle states is encoded by the algebra of field operators. Thus the first relevant question towards possible second-quantized representation of multi D-brane states is this: Is there any natural generalization of the ordinary canonical field algebras which corresponds to the gauge symmetry as the quantum statistical symmetry for D-branes? There is a simple example where we already know one possible answer. That is the old $c = 1$ matrix model, where it is well known that the gauge-invariant Hilbert space is equivalent to the Hilbert space of $N$ free fermions. Recently, essentially the same logic as for the $c = 1$ matrix model is argued to be relevant for a special class of 1/2-BPS operators in $\mathrm{AdS}_5/\mathrm{MSYM}_4$ correspondence, namely in the case of D3-branes.

The *generic* 1/2-BPS operators [3] in the theory are

$$\mathcal{O}_{(k_1,k_2,\ldots,k_n)}(x) \equiv [\mathcal{O}_{k_1}(x)\mathcal{O}_{k_2}(x)\cdots\mathcal{O}_{k_n}(x)]_{(0,k,0)}\ , \tag{2.1}$$

where

$$\mathcal{O}_k(x) \equiv \mathrm{Tr}\left(\phi_{\{i_1}(x)\phi_{i_2}(x)\cdots\phi_{i_k\}}(x)\right) \tag{2.2}$$

is the local product of scalar (hermitean matrix) fields $\phi_i$ ($i = 1, 2, \ldots, 6$). The index $(0, k, 0)$, being the standard Dynkin label with $k = \sum_i k_i$, in the right-hand side indicates to extract the traceless symmetric representation of the $\mathrm{SU}(4) \sim \mathrm{SO}(6)$ R-symmetry group. The dimensions of the representation is given by $Dim(k) = (k+1)(k+2)^2(k+3)/12$. The projection to the symmetric traceless representation is uniquely possible, just by totally symmetrize the tensor indices and subtracting all possible traces. There have been given many arguments showing [3] that, in addition to conformal dimensions, 2- and 3-point functions of the 1/2-BPS operators are not renormalized from free-field results. In the case of the so-called extremal correlators, the nonrenormalization property is argued to be valid for higher-point functions too.

The usual logic [4, 5] for the relevance of a free-fermion representation is summarized as follows. First, let us choose a particular plane, say, 5-6 plane, in the directions transverse to D3-branes and consider only the 'highest-weight' operators of the form

$$\mathcal{O}^J_{(k_1,k_2,\ldots,k_n)}(x)_Z \equiv \mathrm{Tr}\left(Z(x)^{k_1}\right)\mathrm{Tr}\left(Z(x)^{k_2}\right)\cdots\mathrm{Tr}\left(Z(x)^{k_n}\right), \qquad J = k = \sum_i k_i \tag{2.3}$$

with

$$Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)\,. \tag{2.4}$$

The conformal dimensions of these operators satisfy $\Delta = J$ with $J$ being the angular momentum with respect to the 5-6 plane. They form a special class of 1/2-BPS operators. In particular, the single-trace operators ($n = 1$) essentially correspond to ordinary KK gravitons orbiting around the large circle of $S^5$ at the intersection with the 5-6 plane. For relatively large $J$ and $n$ which are comparable to $N$ they have been argued [7] to correspond to giant gravitons [8], spherical D3-branes with dipole-like RR-fields. Of course, we can also consider the conjugates of these operators with opposite angular momentum by replacing $Z$ by $Z^\dagger = \frac{1}{\sqrt{2}}(\phi_5 - i\phi_6)$.

Now, suppose we compute two-point functions

$$\left\langle \overline{\mathcal{O}}^J_{(\ell_1,\ell_2,\ldots,\ell_m)}(x)_{Z^\dagger}\, \mathcal{O}^J_{(k_1,k_2,\ldots,k_n)}(y)_Z \right\rangle . \tag{2.5}$$

Because of the non-renormalization theorem, we can use the (massless) free-field theory with action

$$S_{5,6} = -\frac{1}{2}\int d^4x \;\mathrm{Tr}\left((\partial\phi_5)^2 + (\partial\phi_6)^2\right) = -\int d^4x \;\mathrm{Tr}\left(\partial Z^\dagger \partial Z\right). \tag{2.6}$$

The two-point functions (with $J = k = \ell$, zero otherwise) then take the form

$$\left\langle \overline{\mathcal{O}}^J_{(\ell_1,\ell_2,\ldots,\ell_m)}(x)_{Z^\dagger}\, \mathcal{O}^J_{(k_1,k_2,\ldots,k_n)}(y)_Z \right\rangle = f(\{(k),(\ell)\},N)\,|x-y|^{-2J}\,, \tag{2.7}$$

where the function $f(\{(k),(\ell)\},N)$ is determined by summing over all possible combinations of free-field contractions of the scalar fields between the two operators and hence depends only on $N$ for fixed partitions $\{(k),(\ell)\}$ of the traces and $J$. The space-time factor $|x-y|^{-2J}$ comes from the product of $J$ free propagators. Since the function $f(N)$ is completely independent of the space-time dimensions 4, we can replace the 4-dimensional free field action by the complex harmonic operator in one (Euclidean) dimension,

$$S^Z = \int d\tau \;\mathrm{Tr}\left(\frac{dZ^\dagger(\tau)}{d\tau}\frac{dZ(\tau)}{d\tau} + Z^\dagger(\tau)Z(\tau)\right) \tag{2.8}$$

and simultaneously the two-point functions by

$$\left\langle \overline{\mathcal{O}}^J_{(\ell_1,\ell_2,\ldots,\ell_m)}(\tau_1)\, \mathcal{O}^J_{(k_1,k_2,\ldots,k_n)}(\tau_2) \right\rangle = f(\{(k),(\ell)\},N)\,\mathrm{e}^{-J(\tau_1-\tau_2)}\,. \tag{2.9}$$

The time parameter $\tau$ can be regarded as the radial time of the original 4-dimensional system by making identification

$$\mathrm{e}^{\tau_1-\tau_2} = |x-y|^2\,,$$

which corresponds to a particular foliation of the (euclidean) 4 dimensional base space into $R \times S^3$. Our convention for the 4-dimensional base-space metric is Euclidean. Thus at least for two-point functions of the above type, the 1/2-BPS operators are treated as if they are perturbations on D3-branes which are uniform along the $\mathrm{S}^3$ directions of the base space. Essentially we are dealing with only the spherical fluctuations of D3-branes.

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Since this matrix model contains only one complex matrix $Z$, the above set of two-point functions are represented by free fermions, using the method which is standard in the theory of random matrices. For convenience of the reader, we summarize the main steps of the derivation in the appendix. In view of the fact that the system with a conserved RR-charge of $N$ units is reduced to $N$ free fermions, it is tempting to regard them as the fermions corresponding to $N$ D3-branes. If this identification is correct, the second quantized fermion field from the matrix model should be interpreted as the quantum field of D3-branes. Is this what we are seeking for?

In fact, there are some puzzling features which do not allow us to proceed so straightforwardly. The second-quantized fermion fields corresponding to the above system are defined as

$$\psi(z, z^*) = \sum_{n=0} b_n p_n(z) \mathrm{e}^{-|z|^2} , \tag{2.10}$$

$$\psi(z, z^*)^\dagger = \sum_{n=0} b_n^\dagger p_n(z^*) \mathrm{e}^{-|z|^2} , \tag{2.11}$$

where $p_n(z)$ are the normalized monomials,

$$p_n(z) = \sqrt{\frac{2^n}{\pi n!}} z^n , \qquad \int dz dz^* \mathrm{e}^{-2|z|^2} p_n(z) p_m(z^*) = \delta_{nm} \tag{2.12}$$

and $b_n, b_n^\dagger$ are fermion creation and annihilation operators,

$$\{b_n, b_m^\dagger\} = \delta_{nm} , \qquad b_n|0\rangle = 0 = \langle 0|b_n^\dagger .$$

These field operators satisfy the (lowest Landau level) conditions

$$(z + \frac{\partial}{\partial z^*})\psi(z, z^*) = 0 = (z^* + \frac{\partial}{\partial z})\psi(z, z^*)^\dagger \tag{2.13}$$

reflecting the holomorphic nature of the above special set of 1/2 BPS operators. The traces of the complex matrix $Z$ are represented by fermion bilinears

$$\mathrm{Tr}\left(Z^n\right) \leftrightarrow \int dz dz^* \psi^\dagger(z, z^*) z^n \psi(z, z^*) = \frac{1}{2^{n/2}} \sum_{q=0}^{\infty} \sqrt{\frac{(n+q)!}{q!}} b_{n+q}^\dagger b_q , \tag{2.14}$$

$$\mathrm{Tr}\left((Z^\dagger)^n\right) \leftrightarrow \int dz dz^* \psi^\dagger(z, z^*) (z^*)^n \psi(z, z^*) = \frac{1}{2^{n/2}} \sum_{q=0}^{\infty} \sqrt{\frac{(n+q)!}{q!}} b_q^\dagger b_{n+q} . \tag{2.15}$$

In particular, the hamiltonian is, subtracting the zero-point energy,

$$H = \int dz dz^* \psi(z, z^*)^\dagger \Big( z\frac{\partial}{\partial z} + zz^* \Big) \psi(z, z^*) = \sum_{n=0}^{\infty} n b_n^\dagger b_n .$$

The term $zz^*$ in the braces is necessary to cancel the contribution from the gaussian part of the wave function. The euclidean (Heisenberg) equations of motion are

$$\psi(z, z^*, \tau) = \mathrm{e}^{H\tau} \psi(z, z^*) \mathrm{e}^{-H\tau}, \quad \psi^\dagger(z, z^*, \tau) \equiv \Big(\psi(z, z^*, -\tau)\Big)^\dagger = \mathrm{e}^{H\tau} \psi^\dagger(z, z^*) \mathrm{e}^{-H\tau} . \tag{2.16}$$



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The ground state of $N$ fermions is

$$|N\rangle \equiv b^\dagger_{N-1} b^\dagger_{N-2} \cdots b^\dagger_0 |0\rangle \,. \tag{2.17}$$

The excited states created by acting the above bilinears upon $|N\rangle$ are superpositions of various particle-hole pair states created in the fermi sea of the ground state. In the language of the classification [6] of smooth solultions satisfying the same symmetry and the energy condition $\Delta = J$ on the bulk side, the ground state is nothing but the $\mathrm{AdS}_5 \times S^5$ background itself.

The puzzle is related to the U(1) R-symmetry associated with the angular momentum $J$. The symmetry of the fermion system associated with the angular momentum is the phase transformation

$$z \to e^{i\theta} z, \quad z^* \to e^{-i\theta} z^* \,, \tag{2.18}$$

which is equivalently represented in terms of operator language, assuming that the fields are scalars with respect to SO(6), as $b_n \to \mathrm{e}^{-in\theta} b_n, \quad b^\dagger_n \to e^{in\theta} b^\dagger_n$. Naively, this would require that even the ground state $|N\rangle$ has a nonzero angular momentum $J = \sum_{n=0}^{N-1} 1 = N(N-1)/2$ which is equal to the energy. The corresponding bulk theory actually demands that the energy and angular momentum should be defined relative to the ground state which is the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background itself. This means that the origin of the angular momentum must be redefined depending on the choice of state and on the number of D3-branes. In the case of angular momentum, such a subtraction seems very strange from the viewpoint of field theory of D-branes: ordinarily, a field-theoretical angular-momentum operator has no ambiguity which would require such a subtraction.

Another puzzle related to the above is that we can choose different U(1) directions and then obviously the field operators should be regarded as describing different excitation modes of D3-branes. On the other hand, even if we are treating different excitation modes, the ground state for a given $N$ should be one and the same AdS background itself. But this is not satisfied at least manifestly in the above treatment. We have to identify by hand the ground-states defined on different two-dimensional planes and hence on the different Hilbert spaces, as defining one and the same state. This is very unsatisfactory from our viewpoint pointing towards a possible field theory of D-branes.

To resolve these puzzles, it is desirable to treat all transverse directions of scalar fields on an equal footing. Once it could be achieved, the ground state should be manifestly SO(6) singlet, and general 1/2-BPS operators with arbitrary allowed SO(6) wave functions in the representaion $(0, k, 0)$ should arise as independent excited states on it. Even apart from the above issues, the possibility of extending the fermion description to configurations with multiple U(1) charges is an important question by itself, since then we would be able to describe the situations where multiple giant gravitons are traveling along various different directions in $S^5$ simultaneously. The 1/2-BPS condition can still be satisfied, corresponding to the matrix operators such as

$$w_{i_1 i_2 \cdots i_n} \,\mathrm{Tr}\,(Z_{i_1} Z_{i_2} \cdots) \cdots \mathrm{Tr}\,(\cdots Z_{i_n}) \tag{2.19}$$

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where $w_{i_1 i_2 \cdots i_n}$ is a totally *symmetric* tensor with respect to U(3) group and the scalar matrices are arranged into the complex basis as

$$Z_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\,, \qquad Z_2 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)\,, \qquad Z_3 = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)\,,$$

corresponding to three Cartan directions of SO(6). Note that, if the total symmetrization of $w$-tensors were not imposed and hence other SO(6) representations than $(0, n, 0)$ were assumed, they could lead to lower BPS operators with 1/4 or 1/8 supersymmetries. Note that even this set of operators does not exhaust the whole set of 1/2-BPS operators discussed above, since we could still include the conjugates of these complex matrices by explicitly taking into account the traceless condition with respect to SO(6) indices.[1] From our point of view, it does not seem natural to decouple, as suggested in [5], other directions than the 5-6 plane by introducing an artificial parameter which violates SO(6) symmetry. Of course, it is always possible to introduce symmetry breaking terms for the purpose of studying the system with reduced degrees of freedom[2] after constructing a fully symmetric field theory.



## 3. A generalized exclusion principle and composite fermions

In order to generalize the fermion picture to generic 1/2-BPS operators, it seems at first sight that we have to invent some appropriate generalization of diagonalization technique to the case of many matrices. This has long been one of difficult unsolved problems in matrix field theories. Fortunately, however, in the present case of 1/2-BPS operators, we can justifiably use the free-field approximation for the matrix fields because of various non-renormalization properties [3]. We can compute *exact* correlation functions explicitly at least for two or three-point functions (and some special classes of higher-point functions, such as the so-called extremal correlators) and examine directly whether they allow any interpretation by some generalized fermion picture. We will indeed argue that the essence of the emergence of the fermion picture is *not* in the choice of a special plane.

### 3.1 Factorization theorem and the separation of degrees of freedom

Let us start from considering the properties of correlation functions for generic 1/2-BPS operators. We choose the basis for general 1/2-BPS operators in the form

$$\mathcal{O}^I_{(k_1,k_2,\ldots,k_n)}(x) \equiv w^I_{i_1 \cdots i_k} \operatorname{Tr}\left(\phi_{i_1} \cdots \phi_{i_{k_1}}\right) \cdots \operatorname{Tr}\left(\phi_{i_{k-k_n+1}} \cdots \phi_{i_k}\right), \tag{3.1}$$

where $k = k_1 + k_2 + \cdots k_n$, with $(k_1, \ldots, k_n)$ being the number of matrices in partitioning them into multi-traces, is equal to the conformal dimension $\Delta = k$. Though not orthogonal in the sense of conformal operators, this basis is most convenient for our purpose in this

[1] For a simple example, see the expression (6.1) in the concluding section.

[2] For such a study, though in an entirely different context, see a recent work [9], appearing in the course of completing the present paper, which contains a discussion on a Witten index counting 1/2-BPS states in theories with supersymmetry SU(2|4).

section. If we wish, we can go to the orthogonal basis using the Schur polynomial method as in [4]. For the SO(6) part, $\{w^I_{i_1\cdots i_k}\}$ denotes the basis for totally symmetric traceless tensors. When necessary, one can arrage them so that they satisfy the orthonormality condition $\langle w^{I_1}w^{I_2}\rangle \equiv w^{I_1}_{i_1\cdots i_k}w^{I_2}_{i_1\cdots i_k} = \delta^{I_1I_2}$, defining the $\mathrm{S}^5$ harmonics $Y^I[\phi_i] = w^I_{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$. However, the orthogonality is not necessary for our present purpose.

Now we consider two-point functions using free-field theory. As before, we can actually replace the base space by a one-dimensional space whose coordinate is identified with the radial time ($\tau$) of the original 4 dimensional Euclidean world volume,

$$\left\langle \mathcal{O}^{I_1}_{(k_1,k_2,\ldots,k_n)}(\tau_1)\mathcal{O}^{I_2}_{(\ell_1,\ell_2,\ldots,\ell_n)}(\tau_2)\right\rangle .$$

Because of the traceless condition, the correlator is expressed by all possible contractions of matrix fields between the two *separate* sets of the products of traces of matrix fields at $\tau_1$ and $\tau_2$. A contraction gives the Kronecker delta for the SO(6) indices times the factor $e^{-(\tau_1-\tau_2)}/2$. Because of the total symmetry of the tensors $w^{I_1}$ and $w^{I_2}$, all the different SO(6) contractions associated with these contractions always give one and the same factor $\langle w^I w^J\rangle$. Thus the net result takes a factorized form

$$\left\langle \mathcal{O}^{I_1}_{(k_1,k_2,\ldots,k_n)}(\tau_1)\mathcal{O}^{I_2}_{(\ell_1,\ell_2,\ldots,\ell_n)}(\tau_2)\right\rangle = \langle w^{I_1}w^{I_2}\rangle G(\{(k),(\ell)\},N)e^{-k(\tau_1-\tau_2)} , \tag{3.2}$$

where the factor $G(\{(k),(\ell)\},N)$, which is in fact identical with the previous $f(\{(k),(\ell)\}, N)$, is independent of the SO(6) tensor wave functions and of the coordinate $\tau$. Of course, the factorization of SO(6) invariant $\langle w^{I_1}w^{I_2}\rangle$ itself is just a consequence of the SO(6) symmetry (Wigner-Eckert theorem) which is valid for any theory satisfying a global SO(6) symmetry. What is special to the 1/2 BPS operators for $\mathcal{N}=4$ $\mathrm{SYM}_4$ is that the remaining factor can be replaced by the sum of all possible contractions of free-field theory of a *single hermitean* matrix field. This is valid universally for all partitioning for the matrix traces. Namely, we have

$$G(\{(k),(\ell)\},N)\mathrm{e}^{-r(\tau_1-\tau_2)} = \left\langle :\mathcal{O}^r_{(k_1,\ldots,k_n)}(\tau_1)_M: :\mathcal{O}^r_{(\ell_1,\ldots,\ell_n)}(\tau_2)_M:\right\rangle_M , \tag{3.3}$$

where the matrix operators are defined in the same way as before by replacing the hermitean matrices $\phi_i$ by the single *hermitean* matrix field $M(\tau)$;

$$\mathcal{O}^r_{(k_1,\ldots,k_n)}(\tau_1)_M \equiv \mathrm{Tr}\left(M^{k_1}\right)\mathrm{Tr}\left(M^{k_2}\right)\cdots\mathrm{Tr}\left(M^{k_n}\right) \tag{3.4}$$

with the action $S_M = -\frac{1}{2}\int d\tau\,\mathrm{Tr}(\dot{M}^2+M^2)$. The normal product symbol $:\cdots:$ indicates that no contraction is allowed inside. For the validity of this reduction into the single-matrix model, it is sufficient, because of the free-field approximation, that the SO(6) factors for arbitrary contractions always give a uniquely fixed quantity. This is satisfied also in the case of 3-point functions and for general *extremal* $n$-point functions. In the present section, however, we restrict ourselves only to two-point functions.

What we should learn from these almost trivially looking observations is that the emergence of the one-matrix model for describing 1/2 BPS operators is essentially due to the separation of degrees of freedom into the purely matrix degrees of freedom and the purely kinematical SO(6) degrees of freedom. This is a *dynamical* property which can not be explained by the SO(6) symmetry alone, and whose origin lies in non-renormalization of the 1/2-BPS correlators.[3] The choice of a special plane as in the usual argument is not necessary. The usual manipulation, reviewed in the appendix, is perfectly valid. We are, however, saying that the one-matrix model can play key roles even if we treat all SO(6) directions equivalently. As we see below, this viewpoint provides a natural resolution of our puzzles.

Actually, it is also possible and is more convenient to use a single *complex* matrix field $Z$, instead of the hermitean matrix $M$, since then we can automatically avoid the normal ordering prescription in the case of two-point functions (and also for general extremal correlators). This simply amounts to using a coherent-state representation for one-dimensional (matrix) harmonic oscillator. Thus we have a simple result

$$\left\langle \mathcal{O}^{I_1}_{(k_1,k_2,\dots,k_n)}(\tau_1)\mathcal{O}^{I_2}_{(\ell_1,\ell_2,\dots,\ell_n)}(\tau_2)\right\rangle = \langle w^{I_1}w^{I_2}\rangle \left\langle \overline{\mathcal{O}}^{k}_{(k_1,k_2,\dots,k_n)}(\tau_1)_{Z^\dagger}\mathcal{O}^{k}_{(\ell_1,\ell_2,\dots,\ell_n)}(\tau_2)_Z\right\rangle \tag{3.5}$$

using the same complex matrix model as for the special 1/2-BPS operators satisfying $\Delta = J = k$. If we wish, we can further replace this complex matrix model by the first-order model with the action,

$$S = 2\int d\tau\ \mathrm{Tr}\left[-Z^\dagger\frac{\partial}{\partial\tau}Z + Z^\dagger Z\right]$$

instead of the second order action. The difference between the first and second order models lies only in Green functions. In the former, the propagator is

$$\langle Z^\dagger_{ab}(\tau_1)\, Z_{cd}(\tau_2)\rangle = \begin{cases} \dfrac{1}{2}\mathrm{e}^{-(\tau_1-\tau_2)}\delta_{ad}\delta_{bc} & \tau_1 > \tau_2 \\ \\ 0 & \tau_1 < \tau_2 \end{cases}.$$

Remember that in the case of the previous second order action, the propagator is $\langle Z^\dagger_{ab}(\tau_1)$ $Z_{cd}(\tau_2)\rangle \propto \exp(-|\tau_1-\tau_2|)$. For two-point functions and higher-point extremal correlators, this difference does not matter, since by our definition the operators with $Z^\dagger$'s always appear after those with $Z$'s.

We emphasize that, though we are using the same notation, the meaning of the complex matrix $Z$ is now entirely different from the previous case with the special operators satisfying $\Delta = J$ with a single U(1) charge $J$. Here it is introduced merely as a technical device (coherent-state representation) in order to avoid contractions automatically inside each single matrix operators at a given time. The origin of the lowest Landau level condition is nothing other than the normal ordering condition in (3.3). Therefore, its phase transformation is nothing to do with the angular momentum of a particular SO(2) subgroup of the R-symmetry. Thus, in our formulation, the matrix degrees of freedom are completely inert under SO(6) from the outset. The reason why we were led to interpret the phase rotation (2.18), in the ordinary derivation, as the SO(2) rotation as the subgroup of SO(6) is a

[3]For example, the Wigner-Eckert theorem alone cannot say anything about the relations of invarints, after $\langle w^{I_1}w^{I_2}\rangle$ being factored out, for various *different* configurations of the partitions of matrix traces.



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coincidence of their charges between the SO(6) wave functions $w^I$'s and the wave functions of matrix model: it occurs when we use the complex bases *both* for SO(2) $\subset$ SO(6) and the matrix model. This explains why we had to subtract the charges of the ground state when we reinterpret the phase rotation by the fermion creation-annihilation operators.

Now, our problem is how to interpret these two-point functions by treating generic 1/2-BPS operators with different SO(6) wave functions as *independent* excitation modes of the would-be generalized fermionic fields of D3-branes. Under the motivations explained in the previous section, our task is therefore to

(1) define the D-brane fields which involve coordinate dependence corresponding to all the scalar directions $\phi_i$ on an equal footing;

(2) interpret the generic 1/2-BPS matrix operators as bilinear functions (and their products) in terms of D-brane fields,

such that the above simple factorized form (3.5) of two-point correlation fucntions is obtained.

In the fermion picture, the ground state $|N\rangle$ of $N$ D3-branes is the lowest-energy state which is occupied by $N$ fermions. We assume that the ground states with increasing $N$ are consecutively constructed from ground states with smaller number of fermions by acting creation operators of D3-brane. All these ground states with different $N$ must be SO(6) singlet. Then the corresponding creation operators must also be SO(6) singlet. We denote such SO(6) singlet fermion creation operators by $b^\dagger_{n,0}$ where $n$ labels the energy levels and 0 designates that they are SO(6) singlet. Thus

$$|N\rangle = b^\dagger_{N-1,0} b^\dagger_{N-2,0} \cdots b^\dagger_{0,0}|0\rangle. \tag{3.6}$$

The fermionic nature requires

$$\left\{b^\dagger_{n,0}, b^\dagger_{m,0}\right\} = 0 \tag{3.7}$$

for arbitrary pair of energy levels $(m, n)$. We also introduce the conjugate operators and states

$$\langle N| = \langle 0|b_{0,0} \cdots b_{N-2,0} b_{N-1,0}\,, \tag{3.8}$$

$$\{b_{n,0}, b_{m,0}\} = 0\,, \tag{3.9}$$

satisfying

$$\langle N'|N\rangle = \delta_{N'N}\,. \tag{3.10}$$

Ordinarily, the orthogonality condition is ensured by the standard anticommutation relation of creation and annihilation operators $\{b_{n,0}, b^\dagger_{m,0}\} = \delta_{nm}$ with the vacuum conditions

$$b_{n,0}|0\rangle = 0 = \langle 0|b^\dagger_{m,0}\,. \tag{3.11}$$



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For our purpose here, however, it is important to keep in mind that the anticommution relation between the creation and annhilation operators are not completely compulsory for ensuring the orthogonality. It is sufficient to assume that the annihilation operator indeed annihilates one fermion in the usual manner *when acting upon the gound state*,

$$b_{n,0}|N\rangle = \begin{cases} (-1)^{N-1-n}\, b^\dagger_{N-1,0} b^\dagger_{N-2,0} \cdots \widehat{b^\dagger_{n,0}} \cdots b^\dagger_{0,0}|0\rangle & \text{for } n < N \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

where the object below the hat is absent. We do not assume the operator anticommutation relation $\{b_{n,0}, b^\dagger_{m,0}\} = \delta_{nm}$, and will shortly see that there is still a consistent operator algebra which satisfies the requirement (3.12) and orthogonality condition simultaneously.

### 3.2 'D'-exclusion principle and Cuntz algebra — a generalized Pauli principle for D-branes

Next we have to consider excited states. From now on, we assume for definiteness that the SO(6) basis $w^I$ is orthonormalized, $\langle w^{I_1} w^{I_2}\rangle = \delta^{I_1 I_2}$. As the first step, let us consider an excited state which corresponds to the action of the single trace operator

$$\mathcal{O}^I_{(k)} = w^I_{i_1,i_2,\ldots,i_k}\, \mathrm{Tr}\,(\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k})\,.$$

In the fermion picture, this must correspond to creating a pair of particle and hole in the fermi sea of the ground state in such a way that the created pair has the designated SO(6) state $I$ and lifts the energy by $k$ units (=conformal dimension). The annihilated single-particle state must be one (energy=$n$) of the singlet states occupied in $|N\rangle$ and the created state must then have energy $n+k$ and carry the nontrivial SO(6) index $I$. The two-point function of this operator with its conjugate has to satisfy the above factorization property (3.5). Namely, it must have the $N$ dependence which is identical with the case of the special 1/2-BPS operator with $\Delta = J$, apart from the SO(6) factor. Denoting the creation and annihilation operators with non-singlet index $I$ by $b^\dagger_{n,I}$ and $b_{n,I}$, the relevant part of the corresponding fermion bilinear (its conjugate) would then take the following form

$$2^{-k/2}\sqrt{\frac{(n+k)!}{n!}}\, b^\dagger_{n+k,I} b_{n,0}\,, \qquad 2^{-\ell/2}\sqrt{\frac{(n+\ell)!}{n!}} b^\dagger_{n,0} b_{n+\ell,I'}$$

with the orthogonality condition

$$\langle N|b^\dagger_{n,0} b_{n+\ell,I'}\, b^\dagger_{n+k,I} b_{n,0}|N\rangle = \begin{cases} \delta_{k,\ell}\delta_{I,I'} & \text{for } k+n \geq N, n<N \\ 0 & \text{otherwise} \end{cases}\,. \tag{3.13}$$

The vanishing condition of the second line is one of the crucial and inevitable requirements in our formulation, originating from the factorization property. This means in particular that any two single-particle D3-brane states with the same energy levels must be mutually exclusive even if they have different SO(6) indices,

$$b^\dagger_{n+k,I} b^\dagger_{N-1,0} b^\dagger_{N-2,0} \cdots \widehat{b^\dagger_{n,0}} \cdots b^\dagger_{0,0}|0\rangle = 0 \quad \text{for} \quad n+k<N\,,\ k\neq 0 \quad \text{with aribitrary } I\,. \tag{3.14}$$

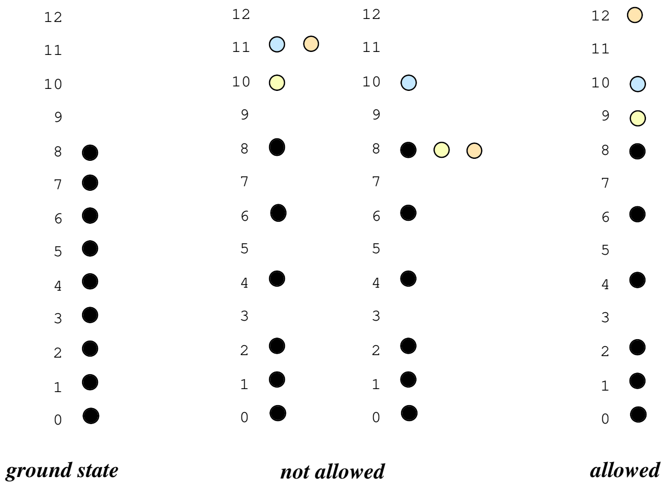


**Figure 1:** This illustrates the Dexclusion principle. The vertical axis indicates the energy levels. The dots are occupied energy levels: black=SO(6) singlet, color(gray) = nonsinglet. The same energy levels cannot be occupied by two or more particles simultaneously, irrespectively of their SO(6) states.

Thus, we are now encountering a stronger version of Pauli's exclusion principle, which should perhaps be interpreted as a signal of the strange quantum statistical property of D-branes represented by gauge symmetry. If the particle with nontrivial SO(6) representation obeys the ordinary Pauli principle that only excludes the case of occupying the same state with completely the same labels with respect to all quantum numbers, the states of particle-hole pairs would have higher degeneracy, since created particles can have energies which are lower than the fermi surface of the ground state. Of course, in this special case of single-trace operators, it is sufficient to assume the above exclusiveness between the singlet and non-singlet representations. We will later see that, to treat multi-trace operators consistently with (3.5), we have to generalize this exclusion property to the cases between arbitrary two SO(6) wave functions, irrespectively of SO(6) representations. We propose to call this generalized exclusion principle for D3-branes, *D-exclusion* (or 'Dexclusion') principle (DEP). See figure 1.[4]

It is not possible to realize the DEP by the usual fermionic algebra. A natural possibility suggested from the separation of degrees of freedom is to assume that the creation and annihilation operators are composites of two independent operators acting in different spaces, each of which carries either the SO(6) vector labels or the energy labels separately as

$$b^{\dagger}_{n,I} = c^{\dagger}_{I} \otimes b^{\dagger}_{n} , \qquad b_{n,I} = c_{I} \otimes b_{n} . \tag{3.15}$$

We call the $b$-type operators 'energy' operators and the $c$-type 'vector' operators. Energies and particle numbers are carried only by the energy operators. For brevity of notation, the product symbol '$\otimes$' will be suppressed below. Since these two kinds of operators are

[4]This should not be confused with the so-called *stringy* exclusion principle [10].

assumed to be mutually commutative, $c_I^\dagger b_n^\dagger = b_n^\dagger c_I^\dagger$, $c_I b_m^\dagger = b_m^\dagger c_I, etc$, the DEP is satisfied

$$b_{n,I}^\dagger b_{n,I'}^\dagger = 0\,, \tag{3.16}$$

providing that the energy creation-annihilation operators satisfy the standard canonical fermion algebra,

$$\{b_n, b_m^\dagger\} = \delta_{n,m}\,, \qquad \{b_n, b_m\} = 0 = \{b_n^\dagger, b_m^\dagger\} \tag{3.17}$$

and the associated vacuum condition $b_n|0\rangle = 0 = \langle 0|b_n^\dagger$. The energy is carried only by the energy operators, and the hamiltonian is simply

$$H = \sum_{n=0}^{\infty} n b_n^\dagger b_n\,, \tag{3.18}$$

resulting the (euclidean) time dependence of the composite operators as $c_I b_n \mathrm{e}^{-n\tau} = c_I b_n(\tau)$ and $c_I^\dagger b_n^\dagger \mathrm{e}^{n\tau} = c_I b_n^\dagger(\tau)$.[5]

Actually, there arises an immediate problem in this proposal. Once we introduce the composite operators as above, there is a danger that the ground states (for $N \geq 2$) (and hence any states) could be infinitely degenerate, even if we require that they are SO(6) singlet. The reason is that there are infinitely many different ways of combining vector creation operators $c_I^\dagger$'s into the singlet representation without any cost of energy. It seems that, at the price of realizing the DEP to reduce the degeneracy of excited states, we are facing the problem of infinite degeneracy of the ground state, and hence an infinitely degenerate Hilbert space. As it stands, this system may not be acceptable as a sensible physical system.

This difficulty can be saved if we can interpret the degeneracy as being due to the existence of an infinite number of superselection sectors. Let us assign charges (that we call 'S'-charges) to these operators by the following phase transformations

$$b_n^\dagger \to \mathrm{e}^{in\theta} b^\dagger\,, \qquad c_I^\dagger \to \mathrm{e}^{-ik(I)} c_I^\dagger \tag{3.19}$$

(similarly for their conjugate operators) where $k(I)$ is the level (=the rank of the traceless symmetric tensors) of the SO(6) state $I$. The original creation (or annihilation) operators of D3-brane have the charges $n - k(I)$ (or $-n + k(I)$),

$$b_{n,I}^\dagger \to \mathrm{e}^{i(n-k(I))\theta} b_{n,I}^\dagger\,.$$

Thus, an excitation of SO(6) representation of higher rank lowers the S-charges. We assume that the multi-particle states must have a definite S charge for a given $N$. The S charge of the ground state $|N\rangle$ is

$$Q_S = \sum_{n=0}^{N-1} 1 = N\frac{(N-1)}{2}$$

[5]The composite fermion operators defined here have some formal resemblance to composite operators which are introduced [11] to describe fractional quantum Hall effect. Though the origin of compositeness is entirely different, it would be interesting if the analogy could be formulated in a more physical way. A different aspect of analogies with fractional quantum Hall effect which may be related to our discussion in section 5 has recently been considered in [12]

which is the largest possible S charge for a given energy $E = N(N-1)/2$. The ground state $|N\rangle$ is uniquely characterized as the lowest-energy state satisfying

$$Q_S = E$$

for a given number $N$ of D3-branes. Below we will construct the bilinear operators corresponding to all 1/2-BPS operators such that they carry zero S-charge. Hence, it is possible to consistently restrict the physical Hilbert space in the super-selection sector of the fixed S charge $Q_S = N(N-1)/2$ for any given $N$. The S-charge is analogous to the U(1) R charge in the case of the special operators with $\Delta = J$. The spacetime interpretation of the S-charge conservation will be discussed later. We will argue that it is related to the scale symmetry as the special case of conformal symmetry of the $\mathcal{N} = 4$ susy Yang-Mills theory, which is apparently lost in moving to 1-dimensional matrix models.

The question is now what is the appropriate algebra for the SO(6) vector operators. An obvious guess would be that they satisfy the standard bosonic algebra. But this would lead to a different normalization $\langle N'|N\rangle = N!$ for the ground state. At first sight, such a deficiency could be trivially circumvented by simply multiplying the normalization factor $1/\sqrt{N!}$. However, it turns out that when we consider general two-point functions of general multi-trace operators it actually leads to a wrong relative normalization for different partitions of the matrix traces, which is not consistent with the fundamental factorization property (3.5) and cannot be removed by the change of overall normalization.

Our proposal is that the vector operators satisfy a special kind of *free* algebra. We introduce the set of operators $c^\dagger_{i_1 i_2 \cdots i_n}, c_{i_1 i_2 \cdots i_n}$ with traceless and totally symmetric SO(6) indices, satisfying

$$c_{i_1 i_2 \cdots i_n} c^\dagger_{i'_1 i'_2 \cdots i'_n} = \delta_{i_1 i_2 \cdots i_n, i'_1 i'_2 \cdots i'_n} \,, \tag{3.20}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} c^\dagger_{i_1 i_2 \cdots i_n} c_{i_1 i_2 \cdots i_n} = 1 \,, \tag{3.21}$$

where $\delta_{i_1 i_2 \cdots i_n, i'_1 i'_2 \cdots i'_n}$ symbolically designates the identity bi-tensor in the space of traceless symmetric tensors of rank $n$. We often denote these tensor operators symbolically as $c_{(n)}, c^\dagger_{(n)}$ suppressing their indices:

$$c_{(k)} c^\dagger_{(\ell)} = \delta_{(k),(\ell)} \,, \qquad \sum_{(k)=0}^{\infty} c_{(k)} c^\dagger_{(k)} = 1 \,.$$

This type of algebras is called the Cuntz algebra [13] in the general theory of $C^*$ algebras. The Cuntz algebra has previously been utilized in field theories for constructing 'master' fields in the large $N$ limit [14]. More recently, it has also been utilized to study the pp-wave limit [15]. All these previous applications are related to the planar limit. In our case, however, $N$ is assumed to be arbitrary and, hence, the role played by the free nature of this algebra in our construction is completely different from those in other applications. Mathematically, it is known [16] that the canonical anticommutative relations (CAR) can be embedded in the Cuntz algebra. This suggests to embed the whole algebra of composite fermions into a larger Cuntz algebra. But we do not pursue such a possiblity in the present work.



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We denote the lowest operator with no SO(6) indices (i.e. identity representation) by $c_0^\dagger$ and $c_0$. The vector operators $c_I, c_I^\dagger$ discussed previously are related to the above by

$$c_I = w^I_{i_1 i_2 \cdots i_n} c_{i_1 i_2 \cdots i_n} \equiv w^I_{(n)} c_{(n)} \,, \qquad c_I^\dagger = w^I_{i_1 i_2 \cdots i_n} c^\dagger_{i_1 i_2 \cdots i_n} \equiv w^I_{(n)} c^\dagger_{(n)} \tag{3.22}$$

which lead to

$$c_I c^\dagger_{I'} = w^I_{i_1 i_2 \cdots i_n} c_{i_1 i_2 \cdots i_n} w^{I'}_{i'_1 i'_2 \cdots i'_n} c^\dagger_{i'_1 i'_2 \cdots i'_m} = \delta_{mn} \langle w^I w^{I'} \rangle = \delta^{II'} \,. \tag{3.23}$$

Although the composite fermion operators themselves do not satisfy the CAR, it is easy to check that the fermionic nature (3.12) of the composite operators when acting upon the ground state, and hence the correct normalization condition, are satisfied,

$$\langle N'|N\rangle = \delta_{N',N} (c_0)^N (c_0^\dagger)^N \langle 0| b_0 \cdots b_{N-2} b_{N-1}\, b^\dagger_{N-1} b^\dagger_{N-2} \cdots b^\dagger_0 |0\rangle = \delta_{N',N}$$

since $(c_0)^N (c_0^\dagger)^N = (c_0)^{N-1} (c_0^\dagger)^{N-1} = \cdots = c_0 c_0^\dagger = 1$. Note that, with respect to the Cuntz algebra, we extract the internal product as the coefficient of the identity. A more legitimate way would be to introduce the Fock vacuum by $c_I|0\rangle_c = 0 = {}_c\langle 0| c_I^\dagger$. Then, we should write as ${}_c\langle 0|(c_0)^N (c_0^\dagger)^N |0\rangle_c = 1$. In this convention, the second definining equation of the Cuntz algebra should be $\sum_I c_I^\dagger c_I = 1 - |0\rangle_c\, {}_c\langle 0|$. As long as we consider only the correlation function for the groud state, however, our convention, following the original form in [13], of not introducing the vacuum state for the Cuntz operators is sufficient and simpler.

## 4. D-brane fields and their bilinears

### 4.1 D-brane fields for spherical D3-branes in 1/2-BPS sector

We are now ready to define the field operators of D3-branes in the 1/2-BPS sector as functions of real coordinates $\phi_i$ and a complex coordinate $\alpha$ (complex conjugate being $\overline{\alpha}$). The fields and their cojugates which annihilate and create spherical D3-branes are, respectively,

$$\Psi_n^{(+)}[\phi, \alpha, \overline{\alpha}] = \begin{cases} \sum_{k=0}^\infty \sqrt{\frac{2^{n+k}}{(n+k)!}} \mathrm{e}^{-|\alpha|^2 - |\phi|^2/4} \frac{f(k)}{\sqrt{k!}} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k} \alpha^{n+k}\, b_{n+k}\, c_{i_1 i_2 \cdots i_k}, & (n \geq 0) \\ \sum_{k=-n}^\infty \sqrt{\frac{2^{n+k}}{(n+k)!}} \mathrm{e}^{-|\alpha|^2 - |\phi|^2/4} \frac{f(k)}{\sqrt{k!}} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k} \alpha^{n+k}\, b_{n+k}\, c_{i_1 i_2 \cdots i_k}, & (n < 0) \end{cases} \tag{4.1}$$

$$\Psi_n^{(-)}[\phi, \alpha, \overline{\alpha}] = \begin{cases} \sum_{k=0}^\infty \sqrt{\frac{2^{n+k}}{(n+k)!}} \mathrm{e}^{-|\alpha|^2 - |\phi|^2/4} \frac{1}{f(k)\sqrt{k!}} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k} \overline{\alpha}^{n+k}\, b^\dagger_{n+k}\, c^\dagger_{i_1 i_2 \cdots i_k}, & (n \geq 0) \\ \sum_{k=0}^\infty \sqrt{\frac{2^{n+k}}{(n+k)!}} \mathrm{e}^{-|\alpha|^2 - |\phi|^2/4} \frac{1}{f(k)\sqrt{k!}} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k} \overline{\alpha}^{n+k}\, b^\dagger_{n+k}\, c^\dagger_{i_1 i_2 \cdots i_k}. & (n < 0) \end{cases} \tag{4.2}$$

We suppressed the time dependence $b_{n+k} = b_{n+k}(\tau), b^\dagger_{n+k} = b^\dagger_{n+k}(\tau)$. These fields are defined such that they have definite S-charges, $Q_S = -n$ and $Q_S = n$, respectively. In view of the ambiguity, $c_{(k)} \to f(k) c_{(k)}, c^\dagger_{(k)} \to c^\dagger_{(k)}/f(k)$, being inherent in the definition of the Cuntz algebra, we temporalily put an undermined $k$-dependent coefficient $f(k)$ which will be fixed later such that it gives the correctly normalized correlation functions. Unless $f(k) = 1$, the fields $\Psi_n^{(+)}$ and $\Psi_n^{(-)}$ are *not* mutually (hermitean) conjugate to each other.

Though the base space involves 8 real coordinates and the time $\tau$, its *effective* spatial dimensions can be regarded as 6, in the sense that the dependence on the six vector coordinates $\phi_i$'s is only through traceless polynomials (i.e. spherical harmonics on $S^5$) and the holomorphy conditions are satisfied with respect to the complex coordinate $\alpha$,

$$\left(\alpha+\frac{\partial}{\partial\overline{\alpha}}\right)\Psi_n^{(+)}=0=\left(\overline{\alpha}+\frac{\partial}{\partial\alpha}\right)\Psi_n^{(-)},\quad \left(\alpha+\frac{\partial}{\partial\overline{\alpha}}\right)\Psi_n^{(-)\dagger}=0=\left(\overline{\alpha}+\frac{\partial}{\partial\alpha}\right)\Psi_n^{(+)\dagger}. \tag{4.3}$$

Roughly, the vector coordinates should be interpreted to be corresponding to $S^5$, while the additional real dimension $x\sim\alpha+\overline{\alpha}$ parametrized by the coherent state representation can be interpreted as the 'radial' direction of $AdS_5$. The apparent duplication of the vector coordinates and the radial coordinate is related to the composite nature of the D-brane creation and annihilation operators. In the direct matrix language, the extraction of the radial direction has been a difficult question. It is interesting that the extended fermion picture indicates a particular way of extracting the radial direction for 1/2-BPS sector. It would be desirable to clarify the spacetime picture in more geometrical terms.

From the viewpoint of ordinary field theories, it seems more natural to define fields by using all of independent composite creation and annilation operators simultaneously as

$$\Psi^{(+)}[\phi,\alpha,\overline{\alpha}]\equiv\sum_{n=-\infty}^{\infty}\Psi_n^{(+)}[\phi,\alpha,\overline{\alpha}]\,,\qquad \Psi^{(-)}[\phi,\alpha,\overline{\alpha}]\equiv\sum_{n=-\infty}^{\infty}\Psi_n^{(-)}[\phi,\alpha,\overline{\alpha}]\,. \tag{4.4}$$

Neither of the fields $\Psi_n^{(+)},\Psi_n^{(-)}$ with definite S-charge, nor the $\Psi^{(+)},\Psi^{(-)}$ without definite S-charge, do not satisfy the standard canonical (anti-) commutation relations, because of the free nature of the Cuntz algebra. Hence, they are not local in the sense of the usual framework of quantum field theory.

However, the fields $\Psi^{(+)},\Psi^{(-)}$ without definite S-charge can be regarded as a sort of local fields which are mutually 'quasi-canonical' conjugate to each other. This can be seen by looking at the effect of their action on the vaccum:

$$\Psi^{(-)}[\phi,\alpha,\overline{\alpha}]\Psi^{(+)}[\phi',\alpha',\overline{\alpha'}]|0\rangle=|0\rangle\,\delta[\phi,\alpha,\overline{\alpha};\phi',\alpha',\overline{\alpha}'] \tag{4.5}$$

where

$$\delta[\phi,\alpha,\overline{\alpha};\phi',\alpha',\overline{\alpha}']=\delta[\phi;\phi']\,\delta[\alpha,\overline{\alpha};\alpha',\overline{\alpha}']\,, \tag{4.6}$$

$$\delta[\phi;\phi']\equiv\sum_{(k)=0}^{\infty}\frac{1}{k!}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}\phi'_{i_1}\phi'_{i_2}\cdots\phi'_{i_k}\,\mathrm{e}^{-(|\phi|^2+|\phi'|^2)/4}\,, \tag{4.7}$$

$$\delta[\alpha,\overline{\alpha};\alpha',\overline{\alpha}']\equiv\sum_{n=0}^{\infty}\frac{2^n}{n!}\left(\overline{\alpha}\alpha'\right)^n\mathrm{e}^{-|\alpha|^2-|\alpha'|^2}\,, \tag{4.8}$$

are delta-functions in the spaces of wave functions of the vector and radial (holomorphic) coordinates, respectively, satisfying

$$\int[d\phi']G[\phi']\,\mathrm{e}^{-|\phi|^2/4}\delta[\phi;\phi']=G[\phi]\,\mathrm{e}^{-|\phi|^2/4}\,, \tag{4.9}$$

$$\int d\alpha' d\overline{\alpha}' F(\alpha')\,\mathrm{e}^{-|\alpha'|^2}\delta[\alpha,\overline{\alpha};\alpha',\overline{\alpha}']=F(\alpha)\,\mathrm{e}^{-|\alpha|^2}\,, \tag{4.10}$$



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for arbitraty polynomial functions, $G[\phi]$ and $F[\alpha]$. The integration measure in the space of single-particle wave functions is normalized such that

$$\int [d^6\phi|d\alpha|^2]\exp(-2\alpha\overline{\alpha}-|\phi|^2/2)=1\,, \qquad (|\phi|^2=\sum_{i=1}^{6}\phi_i^2)\,. \tag{4.11}$$

The origin of the gaussian measure is of course the free-field nature of the matrix model. Thus, the fields $\Psi^{(+)},\Psi^{(-)}$ correspond to one-particle states which can be strictly localized in the base space. However, this in turn implies that, in any allowed quantum state with a *definite* value of S-charge, D3-branes cannot in general be localized with respect to the transverse directions. Note also that the fields and their hermitean conjugates cannot be local with respect to each other in any sense unless $f(k)=1$.

### 4.2 Bilinears

Since we demand that the S-charge is a superselection charge, the allowed observables in our Hilbert space must have zero S-charge. Under this convention, we introduce the integrated bilinear operator which corresponds to a single trace operator $w^I_{i_1i_2\cdots i_k}\,\mathrm{Tr}(\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k})$,

$$\int [d^6\phi|d\alpha|^2]\sum_{n=-\infty}^{\infty}\Psi_n^{(-)}[\phi,\alpha,\overline{\alpha}]w^I_{i_1i_2\cdots i_k}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}\alpha^k\Psi_n^{(+)}[\phi,\alpha,\overline{\alpha}]= \tag{4.12}$$
$$=2^{-k/2}\sum_{n+k_2\geq 0}^{\infty}\frac{\sqrt{(n+k+k_2)!}}{\sqrt{(n+k_2)!}}\frac{f(k_2)\sqrt{k_1!}}{f(k_1)\sqrt{k_2!}}\langle c^\dagger_{(k_1)}w^I_{(k)}c_{(k_2)}\rangle b^\dagger_{n+k_1}b_{n+k_2}\,,\quad (k_1=k+k_2)\,.$$

This is appropriate when the matrix operator corresponds to the holomorphic matrix operator in the factorized expression using the complex representation (3.5). When the operator corresponds to anti-holomorphic matrix operator, this has to be replaced by its hermitean conjugate,

$$\int [d^6\phi|d\alpha|^2]\sum_{n=-\infty}^{\infty}\Psi_n^{(+)}[\phi,\alpha,\overline{\alpha}]^\dagger w^I_{i_1i_2\cdots i_k}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}\overline{\alpha}^k\Psi_n^{(-)}[\phi,\alpha,\overline{\alpha}]^\dagger= \tag{4.13}$$
$$=2^{-k/2}\sum_{n+k_1\geq 0}^{\infty}\frac{\sqrt{(n+k+k_1)!}}{\sqrt{(n+k_1)!}}\frac{f(k_1)\sqrt{k_2!}}{f(k_2)\sqrt{k_1!}}\langle c^\dagger_{(k_1)}w^I_{(k)}c_{(k_2)}\rangle b^\dagger_{n+k_1}b_{n+k_2}\,,\quad (k_2=k+k_1)\,.$$

It will turn out that for the correct normalization of correlators the choice

$$f(k)=\sqrt{k!} \tag{4.14}$$

is most appropriate, such that the fermion part of the bilinear operators take the same form as those obtained from the comlex 1-matrix model and hence the factorization property of the correlators are faithfully reallized.

These purely holomorphic and anti-holomorphic operators are sufficient to compute all correlators of the extremal type which include the two-point functions. In the non-extremal case, the identification of (4.13) or its conjugate (4.14) with the single-trace matrix operator $w^I_{i_1 i_2\cdots i_k}\,\mathrm{Tr}(\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k})$ is not sufficient for properly taking into account the normal ordering prescription, since then the bilinear operators of mixed type become necessary. In this first work, we restrict ourselves only to the extremal case.

As a special case of general bilinear operators that correspond to the case $k=0$, the hamiltonian is expressed as

$$
\begin{aligned}
H &= \int [d^6\phi |d\alpha|^2] \sum_{n=-\infty}^{\infty} \Psi_n^{(-)}[\phi,\alpha,\overline{\alpha}] \left(\alpha \frac{\partial}{\partial \alpha} + \alpha\overline{\alpha}\right) \Psi_n^{(+)}[\phi,\alpha,\overline{\alpha}] \\
&= \Big(\sum_{(k)=0}^{\infty} c^\dagger_{(k)} c_{(k)}\Big)\Big(\sum_{n=0}^{\infty} n b^\dagger_n b_n\Big) = \sum_{n=0}^{\infty} n b^\dagger_n b_n\,, \qquad (4.15)
\end{aligned}
$$

which is automatically hermitean for arbitrary $f(k)$. We note that actually the hamiltonian can also be expressed in the standard local form using the fields without definite S-charge as

$$
H = \int [d^6\phi |d\alpha|^2] \Psi^{(-)}[\phi,\alpha,\overline{\alpha}] \left(\alpha \frac{\partial}{\partial \alpha} + \alpha\overline{\alpha}\right) \Psi^{(+)}[\phi,\alpha,\overline{\alpha}]\,. \qquad (4.16)
$$

Since this behaves as the identity with respect to the Cuntz algebra, the Heisenberg equation of motion is consistent with the time dependence that has been assumed in the foregoing discussions. Similarly, the number operator is

$$
\begin{aligned}
N &= \int [d^6\phi |d\alpha|^2] \sum_{n=-\infty}^{\infty} \Psi_n^{(-)}[\phi,\alpha,\overline{\alpha}] \Psi_n^{(+)}[\phi,\alpha,\overline{\alpha}] \\
&= \Big(\sum_{(k)=0}^{\infty} c^\dagger_{(k)} c_{(k)}\Big)\Big(\sum_{n=0}^{\infty} b^\dagger_n b_n\Big) = \sum_{n=0}^{\infty} b^\dagger_n b_n. \qquad (4.17)
\end{aligned}
$$

This can also be expressed in the local form in terms of $\Psi^{(+)}, \Psi^{(-)}$.

Finally, for purely 'kinematical' operators as the bilinears of the Cuntz operators alone are represented as *ratios* of the ordinary bilinears. For instance, we can define

$$
R \equiv \sum_{k=0}^{\infty} k\, c^\dagger_{(k)} c_{(k)} = N^{-1} \int [d^6\phi |d\alpha|^2] \Psi^{(-)}[\phi,\alpha,\overline{\alpha}] \Big(\phi_i \frac{\partial}{\partial \phi_i} + \frac{1}{2}|\phi_i|^2\Big) \Psi^{(+)}[\phi,\alpha,\overline{\alpha}], \quad (4.18)
$$

where the expression $N^{-1}$ is meant that it acts upon arbitrary 1/2-BPS states except for the Fock vacuum. This operator counts the number of transverse scalar fields $\phi_i$ and satisfies

$$
R\Psi^{(-)} = \Big(\phi_i \frac{\partial}{\partial \phi_i} + \frac{1}{2}|\phi_i|^2\Big)\Psi^{(-)}\,, \qquad \Psi^{(+)} R = \Big(\phi_i \frac{\partial}{\partial \phi_i} + \frac{1}{2}|\phi_i|^2\Big)\Psi^{(+)}\,. \qquad (4.19)
$$

The S-charge operator is then given by

$$
Q_S = H - R\,, \qquad (4.20)
$$

whose commutator with a bilinear counts its S-charge. It is therefore commutative with the above (4.13) and (4.14). Note that to compute the commutator of R and general bilinears, the above two relations are sufficient: $[R, c^\dagger_{k_1} c_{k_2}] = (k_1 - k_2) c^\dagger_{k_1} c_{k_2}$. We can



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similarly construct SO(6) generators as,

$$J_{ij} \equiv N^{-1} \int [d^6\phi |d\alpha|^2] \Psi^{(-)}[\phi, \alpha, \overline{\alpha}] \Big( \phi_i \frac{\partial}{\partial \phi_j} - \phi_j \frac{\partial}{\partial \phi_i} \Big) \Psi^{(+)}[\phi, \alpha, \overline{\alpha}], \tag{4.21}$$

satisfying

$$J_{ij}\Psi^{(-)} = -\Big( \phi_i \frac{\partial}{\partial \phi_j} - \phi_j \frac{\partial}{\partial \phi_i} \Big) \Psi^{(-)}, \qquad \Psi^{(+)} J_{ij} = \Big( \phi_i \frac{\partial}{\partial \phi_j} - \phi_j \frac{\partial}{\partial \phi_i} \Big) \Psi^{(+)}. \tag{4.22}$$

The SO(6) transformations of bilinears are given by taking commutator with $J_{ij}$.

### 4.3 Two-point functions

Let us now check that these S-charge invariant bilinear operators give correct correlation functions. In what follows, we always assume that the operators are time-ordered with respect to the Euclidean time $\tau$. For two-point functions of single-trace operators,

$$\Big\langle w^{I_1}_{i_1 i_2 \cdots i_k} \mathrm{Tr}(\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k})(\tau_1)\, w^{I_2}_{j_1 i_2 \cdots j_k} \mathrm{Tr}(\phi_{j_1}\phi_{j_2}\cdots\phi_{j_k})(\tau_2) \Big\rangle ,$$

only the terms with $k_2 = 0, k_1 = k$ or $k_1 = 0, k_2 = k$ of the above bilinear operators contribute, since the action of the vector creation or annihilation operators on the ground state is nonvanishing only for the trivial representation. Here we recover the time dependence, $\exp(-k(\tau_1 - \tau_2))$. Thus the extended fermion representation of the two-point function with the choice (4.14) is equal to

$$\begin{aligned} &\langle N | \langle c_0^\dagger w^{I_1}_{(k)} c_{(k)} \rangle \langle c^\dagger_{(k)} w^{I_2}_{(k)} c_0 \rangle 2^{-k} \sum_n \sqrt{\frac{(n+k)!}{n!}} b_n^\dagger b_{n+k} \sum_m \sqrt{\frac{(m+k)!}{m!}} b^\dagger_{m+k} b_m |N\rangle \times \\ &\times \exp\Big( -k(\tau_1 - \tau_2) \Big) = \\ &= 2^{-k} \langle w^{I_1}_{(k)} w^{I_2}_{(k)} \rangle \langle N | \sum_n \sqrt{\frac{(n+k)!}{n!}} b_n^\dagger b_{n+k} \sum_m \sqrt{\frac{(m+k)!}{m!}} b^\dagger_{m+k} b_m |N\rangle \exp\left(-k(\tau_1 - \tau_2)\right). \end{aligned} \tag{4.23}$$

This is precisely the required form satisfying the factorization property (3.5).

For the multi-trace operator $\mathcal{O}^I_{(k_1,k_2,\ldots,k_n)}$, the corresponding operator acting upon the ground state is, when it corresponds to the holomorphic matrix operator,

$$w^I_{(k_1+k_2+\cdots+k_n)} B_{(k_1)} B_{(k_2)} \cdots B_{(k_n)}, \quad (k_1 + k_2 + \cdots + k_n = k) \tag{4.24}$$

where $B_{(k_a)}$'s are the bilinear operators with completely symmetrized tensor indices of rank $k_i$,

$$B_{(k_a)} = \int [d^6\phi |d\alpha|^2] \sum_{n=-\infty}^{\infty} \Psi_n^{(-)}[\phi, \alpha, \overline{\alpha}] \phi_{(i_1}\phi_{i_2}\cdots\phi_{i_{k_a})} \alpha^k \Psi_n^{(+)}[\phi, \alpha, \overline{\alpha}]. \tag{4.25}$$

Remember that in the above expressions the tensor indices are symbolically represened by the lower subscript such as $(k)$. Since the indices are contracted with $w^I$, $B_{(k_a)}$ takes the same form as (4.13) by replacing the $w$-tensor in the latter by the part of tensor indices

$(k_a)$. If this string of the bilinears acts upon the ground state, each factor shifts the rank of the vector operators successively as $0 \to k_n \to k_n + k_{n-1} \to \cdots \to k_n + k_{n-1} + \cdots + k_1 = k$ (from right to left). Other types of shiftings do not contribute to the extremal correlators. In this way, we arrive at the following product of fermion bilinears that acts upon the ground state,

$$
\begin{aligned}
&2^{-k/2} \langle c^\dagger_{(k)} w^I_{(k)} c_0 \rangle \sum_{\ell_1=0}^{\infty} \sqrt{\frac{(\ell_1 + k_1)!}{\ell_1!}} b^\dagger_{\ell_1+k_1} b_{\ell_1} \cdots \times \qquad (4.26)\\
&\times \cdots \sum_{\ell_{n-1}=0}^{\infty} \sqrt{\frac{(\ell_{n-1} + k_{n-1})!}{\ell_{n-1}!}} b^\dagger_{\ell_{n-1}+k_{n-1}} b_{\ell_{n-1}} \sum_{\ell_n=0}^{\infty} \sqrt{\frac{(\ell_n + k_n)!}{\ell_n!}} b^\dagger_{\ell_n+k_n} b_{\ell_n} |N\rangle \, \mathrm{e}^{k\tau_2} \,.
\end{aligned}
$$

Although the field operators do not satisfy any simple commutation relations among themselves, their bilinears $B_{(k_i)}$ give a commutative algebra after acting upon the ground state.

Similarly, we obtain the conjugate operator acting to the left upon $\langle N|$ with time dependent factor $\mathrm{e}^{-k\tau_1}$. This result is valid for arbitrary partition of the traceless symmetric tensor indices of the $w$-tensors. They give the required factorized expressions of the form (3.5) with correct normalization for arbitrary two-point functions of multi-trace operators.

To obtain the correctly normalized expressions, the Cuntz algebra $c_{(k)} c^\dagger_{(\ell)} = \delta_{(k),(\ell)}$ is crucial. If we assumed the usual bosonic algebra, the matrix operators with different partitions of the vector indices would have given differently normalized expressions for the product of fermion bilinears, depending on the manner of partitions. That would not be compatible with the factorization property (3.5). In other words, they would reduce to wrongly normalized products of fermion bilinears even when we consider the special 1/2-BPS operators with a single charge satisfying the condition $\Delta = J$. The origin of such discrepancy is the same as the one which leads to the wrong normalization of ground state $|N\rangle$, as we have mentioned previously.

### 4.4 Higher-point extremal correlators

The case of higher-point extremal correlators is essentially understood from the structure of two-point functions. The term 'extremal' means that the conformal dimensions of the operators satisfy $\Delta_1 = \Delta_2 + \Delta_3 + \cdots + \Delta_n$. The free-field contractions obviously give the following structure

$$
\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_2) \rangle = \langle w_1 w_2 \cdots w_n \rangle \, G(\{(1), (2), \ldots, (n)\}; N) \prod_{A=2}^{n} \frac{1}{|x_1 - x_A|^{2\Delta_A}} \,. \qquad (4.27)
$$

It has been cojectured, with ample evidence just as for 2- and 3-point correlators, that this form is not renormalized from free-field result.

As in the case of two-point correlators, the factorization into 1-single matrix models owes to the uniqueness of the SO(6) invariant. The function $G(\{(1), (2), \ldots, (n)\}; N)$ is again given by the one-matrix hermitean model, replacing the SO(6) matrices $\phi_i$ by the single 1-dimensional hermitean matrix field $M(\tau)$, with normal ordering condition being



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understood in each operator.

$$\langle :\mathcal{O}_1(\tau_1)::\mathcal{O}_2(\tau_2):\cdots:\mathcal{O}_n(\tau_2):\rangle_M = G(\{(1),(2),\ldots,(n)\};N)\prod_{A=2}^{n}\mathrm{e}^{-\Delta_A(\tau_1-\tau_A)}\,. \qquad (4.28)$$

As before, the correspondence of the spacetime factor is given by $\mathrm{e}^{\tau_1-\tau_A} \leftrightarrow |x_1 - x_A|^2$.

Here, we have assumed, without loss of generality, that the operator $\mathcal{O}_1(\tau_1) \leftrightarrow \mathcal{O}_1(x_1)$ is located at the largest time ($\tau_1 > \tau_2 > \cdots > \tau_n$) by choosing the origin of the base spacetime coordinates of the super Yang-Mills theory appropriately. Then, the normal-ordering prescription is handled by converting to the complex 1-matrix model in the same way as for the two-point case: all of the operators except for the first one $\mathcal{O}_1(\tau_1)$ is consisting of the holomorphic matrix $Z$, while the first one is chosen to be the anti-holomorphic operator consisting of the conjugate matrix $Z^\dagger$. Other time-orderings are of course possible, but handling of the normal ordering prescription would become more cumbersome. The situation is now almost the same as in the two-point case. Only difference is the time dependence, resulting in the correct factor $\prod_{A=2}^{n}\mathrm{e}^{-\Delta_A(\tau_1-\tau_A)}$. It is now clear that the extremal correlators can be reproduced by our generalized fermion picture in the same way as in the two-point functions.

**A short comment on 3-point non-extremal case.** There are good reasons to believe (see [3]) that the nonrenormalization property is valid for 3-point functions including non-extremal cases. Therefore it is natural to try to extend our formalism to 3-point functions. In fact, it is not so difficult to do so for some simple cases. One complication is that the normal ordering prescription requires more complicated procedures. For non-extremal 3-point functions, we have to introduce 'mixed' bilinear operators, being located at the middle time, which have dependence on both $\alpha$ and $\overline{\alpha}$, when the bilinear operators are not directly acting upon the ground state. For such operators, we have to explicitly subtract the contributions from contractions inside the matrix operators. In spite of this, it is possible to show that the OPE coefficients can be correctly reproduced by extending the present approach.

Another problem which is more fundamental is that our 'spherical' approximation in which the space-time dependence of correlators is obtained by a simple substitution $\mathrm{e}^{|\tau_1-\tau_2|} \to |x_1 - x_2|^2$ is no more sufficient to reproduce the spacetime dependence of non-extremal 3-point functions. To give a satisfactory treatment, it is therefore necessary to extend our formalism such that non-spherical modes, with respect to the base space of D3-branes, are taken into account appropriately. We postpone such elaborations to later works.

### 4.5 The meaning of the S-charge symmetry

Let us here discuss the spacetime meaning of the S-charge superselection rule. The product of bilinear operators in the expression (4.26) is, by construction, invariant under the S-charge transformation

$$c^\dagger_{(k)} \to \mathrm{e}^{-ik\theta}c^\dagger_{(k)}, \quad c_{(k)} \to e^{ik\theta}c_{(k)}, \quad b^\dagger_n \to \mathrm{e}^{in\theta}b^\dagger_n, \quad b_n \to \mathrm{e}^{-in\theta}b_n$$

which is generated by the operator $Q_S = H - R$. If one 'Wick'-rotates the angle, $\theta \to i\sigma$ ($\sigma$ = real), this is equivalent to the scaling that transforms simultaneously the transverse vectors $\phi_i$'s and the time variable as

$$\phi_i \to \lambda\phi_i\,, \qquad \mathrm{e}^{\tau_2} \to \lambda^{-1}e^{\tau_2} \qquad (\lambda \equiv e^{\sigma})\,, \tag{4.29}$$

*when* one expresses the transformation in terms of the original matrix field variables. For the conjugate operator, the transformation is expressed as

$$\phi_i \to \lambda\phi_i\,, \quad \mathrm{e}^{-\tau_1} \to \lambda^{-1}e^{-\tau_1} \qquad (\lambda \equiv e^{-\sigma})\,. \tag{4.30}$$

Thus, the states constructed by acting the bilinears, and hence the correlation functions, must be invariant under these scalings. This is precisely equivalent to the scaling symmetry of two-point (and general extremal) correlation functions, since in terms of the original base spacetime coordinates $x_i$ $(i = 1, 2)$ of the super Yang-Mills theory,

$$\mathrm{e}^{\tau_1-\tau_2} = |x_1 - x_2|^2 \to \lambda^{-2}\mathrm{e}^{\tau_1-\tau_2} = \lambda^{-2}|x_1 - x_2|^2\,.$$

As argued in [17], behind the (generalized) scaling symmetries, a dual uncertainty relation of spacetime [18] characterizes the dynamics of D-branes: $\Delta T\Delta X \gtrsim \alpha'$ with $\Delta T$ being the typical microscopic scale in the longitudinal direction $\sim x^\mu$ along branes, and $\Delta X$ being the scale in the transverse directions, $X \sim \phi_i$. The scale transformation $\Delta T \to \lambda^{-1}\Delta T, \Delta X \to \lambda\Delta X$ changes these two characteristic scales, but the changes are consistent with the existence of an uncertainty relation of the above type. As emphasized in the first part of the present section, the S-charge superselection rule leads to a certain nonlocality with respect to six effective spatial dimensions. Since the nonlocality is not directly characterized by the form of wave functions, which are ordinary functions of the coordinates as in the usual local field theory, nor by the form of the hamiltonian which is a local bilinear function, the above characterization by the spacetime uncertainty relation seems quite appropriate. Of course, the precise nature must further be clarified.

## 5. Multi-charge geometries: superstar and its entropy

### 5.1 Dexclusion principle and holographay

From the viewpoint of AdS/CFT correspondence, our discussions so far are from the side of the boundary CFT theory, to the extent that we have been trying to construct a second-quantized theory of super Yang-Mills theory in the 1/2-BPS sector. One of the characteristics of our construction is that the quantum statistics of D-brane creation and annihilation operators must be assumed to satisfy an extended version of Pauli principle. Thus, a natural question is now what is the interpretation of the Dexclusion principle from the standpoint of bulk supergravity theory.

If we choose different directions in breaking SO(6) correponding to a partcular angular momentum $J$, the two-dimensional plane of the droplet in the LLM classification is deformed by various SO(6) rotations. Generic solutions generated by those SO(6) rotations

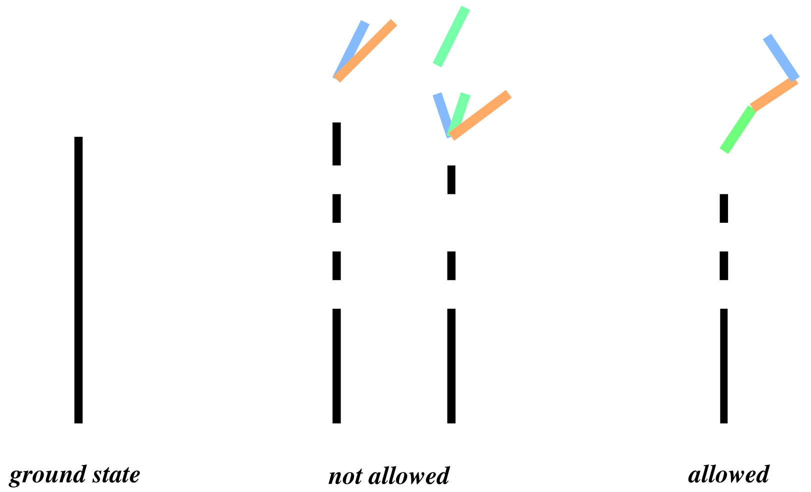


**Figure 2:** The Dexclusion principle from the bulk viewpoint: The droplets are in general deformed when particles are excited with non-singlet SO(6) wave functions, but must have always the fixed constant density with a single sheet. They cannot bifurcate into two or more sheets. The lines show sections of droplet seen when cut by an appropriate plane intersecting with the droplets: black=singlet, color (gray)=nonsinglet. Compare with figure 1.

have nonzero charges with respect to three different SO(2) directions. However, the basic property of droplet that it is incompressible is preserved by such deformations, so that the density of the droplets must always be constant with the same density as in the case of single charge solutions. The sheets extending into different directions in the bulk would correspond to different SO(6) states of (giant) graviton configurations. This implies that the two-dimesional sheet of the droplet would always be consisting of one sheet. In other words, the sheet of the droplet would not bifurcate into two or more sheets.

That this is a natural bulk correspondent of the Dexclusion property can be easily understood by imagining a situtation where the DEP is not satisfied and hence we have excitations of two different SO(6) states simultaneously with the same energy. Then, the corresponding situation on the bulk side would look like this: the sheet defining the ground state is deformed, after this excitation, into a configuration which bifurcates somewhere near the boundary of the ground-state droplet into two independent sheets extending in different directions. See figure 2 in comparison with figure 1. In fact, it is very difficult to imagine that the set of smooth classical solutions exhibit such a singular behavior as bifurcation. Thus, from the viewpoint of the holographic correspondence between bulk and boundary theories, the Dexclusion principle seems to be a natural extension of the usual Pauli principle in formulating D-brane field theory.

It would be very interesting to formulate this qualitative picture in a more concrete form on the bulk side by extending the analysis of LLM [6] to more general configurations with smaller isometries, so that we could treat 'covariantly' all of 1/2-BPS SO(6) states with multiple SO(2) charges. It would require us to extend their ansatz in such a way that, with respect to the isometry SO(4) $\times$ $SO(4)$ $\times$ $\mathbb{R}$, only the first SO(4) isometry corresponding to the spherical approximation of D3-branes is retained while the remaining

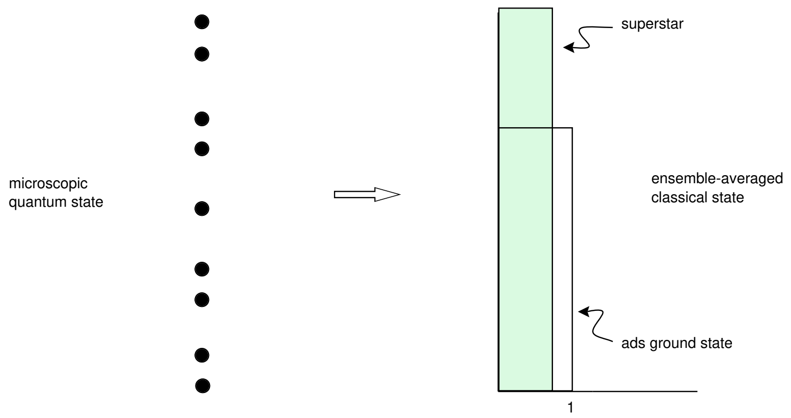


**Figure 3:** The microstates (left) of a superstar and the corresponding averaged configuration (right) represented by a droplet with a lower density.

factors are replaced by some appropriate form which allows inclusion of all possible SO(6) states corresponding to the representation $(0, k, 0)$.

### 5.2 Superstar entropy

At present, only solutions whose form is explicitly known with a more general SO(6) configurations are the so-called 'superstar' solutions [19]. In the extremal limit, they in general have (naked) singularities at their center. It has been shown that they have some characteristics which can be interpreted in terms of the condensation of giant gravitons. It is interesting, as is already noted by LLM, that in the single charge case satisfying the condition $\Delta = J$, a superstar corresponds to a choice of droplet with a density lower than the smooth solutions. The lower density of a superstar droplet could be intepreted as a statistical average of loosely packed occupied states[6] of fermions. There exist a large number of nonsingular solutions with the same charge and the same mass as a singular superstar solution, but with different microscopic distributions of occupied states. Although the existence of both nonsingular and singular solutions with the same charges is mysterious and hence precise relation between the nonsingular and singular solutions must be clarified, it seems natural to view a superstar as an approximation to the ensemble of a large number of droplet configurations corresponding to nonsingular LLM solutions, as in figure 3. The apparent singularity of the superstar solution would be smoothed out by stringy quantum effects, as observed in other situations [20]. The entropy of a general superstar would then be obtained by counting microstates corresponding to the nonsingular LLM solutions [21, 24]. Let us make a modest consistency check on the entropy of superstar from this viewpoint. This viewpoint is in accord with the so-called 'Fuzzball' conjecture [22]. Similar ideas as above for superstars have recently been discussed for the case of D1-D5 system in the literature [23]. Even Schwarzschild-type black holes without any supersymmetry might be understood in a similar way [24].

[6]This is reminiscent of the situation in fractional quantum Hall effect, to which an analogy was already alluded in section 3 from a different aspect.

The generic superstar solutions that have smaller symmetries than $\mathrm{SO}(4)\times\mathrm{SO}(4)\times\mathbb{R}$ have three independent U(1) charges $(J_1, J_2, J_3)$ corresponding to $\mathrm{U}(1)^3$ embedded in the R-symmetry group $\mathrm{SU}(4)\sim\mathrm{SO}(6)$. The energy (mass) equals to the sum

$$\Delta = |J_1| + |J_2| + |J_3|\,,$$

which might naively be interpreted to be an indication that the microscopic degrees of freedom could be independently distributed over the 3 directions of $\mathrm{U}(1)^3$. According to our extended fermion field theory, we can make a simple prediction for the entropy of generic superstar. The generic quantum states take the factorized form (4.26) for a given SO(6) wave function. Therefore, provided that the entropy of the single charge superstar is given as

$$S_{\mathrm{single}} = f(|J|)\,,$$

the entropy of three charge superstar takes the form

$$S_{\mathrm{multi}} = f(|J_1| + |J_2| + |J_3|)\,, \tag{5.1}$$

using the same function $f(J)$. This itself is a consequence of SO(6) symmetry since the entropy must be invariant under SO(6) and $k \equiv |J_1| + |J_2| + |J_3|$ is the Dynkin label specifying the SO(6) representation. However, the Dexclusion pricniple (DEP) shows that the particles with different charges cannot be excited independently, and therefore the number of microstates must be smaller than the case of independent excitations. This means that the inequality

$$f(|J_1| + |J_2| + |J_3|) < f(|J_1|) + f(|J_2|) + f(|J_3|) \tag{5.2}$$

must be satisfied when at least two of the angular momenta are not zero. If we have assumed naively that the mircoscopic degrees of freedom were distributed independently for different directions, (5.2) must be replaced by equality, and hence we would be lead to a *wrong* prediction that $f(J)\propto J$.

Consider the partition function of fermion spectrum $(u, v < 1)$

$$Z(u,v) = \mathrm{Tr}\left(u^N v^H\right) = \sum_{n=0}^{\infty} d_n(v) u^n = \sum_{n,m=0} d_{n,m} v^{m+\frac{n(n-1)}{2}} u^n\,. \tag{5.3}$$

The entropy of the superstar with the total $\mathrm{U}(1)^3$ charge $\Delta$ and total RR-charge $N$ is given as

$$S = \ln d_{N,\Delta}\,. \tag{5.4}$$

Using the product formula

$$\prod_{n=0}^{K}(1+uv^n) = 1 + \sum_{r=1}^{K+1} \frac{(1-v^{K+1})(1-v^K)\cdots(1-v^{K+1-r+1})}{(1-v)(1-v^2)\cdots(1-v^r)} v^{r(r-1)/2} u^r\,,$$

we find, after taking the limit $K\to\infty$,

$$d_n(v) = v^{n(n-1)/2} \prod_{r=1}^{n} \frac{1}{1-v^r}\,, \tag{5.5}$$

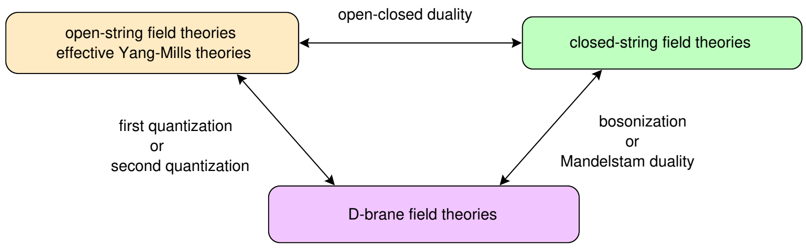


**Figure 4:** This illustates the conceptual relation of a possible framework of D-brane field theory with the usual open and closed string-field theories.

which can also be directly derived from the matrix representation in the single-charge case. This enable us to estimate the entropy in the limit of large $N$ and $\Delta$ satisfying $\Delta/N^2 \ll 1, N \gg 1, \Delta \gg 1$, as [21]

$$f(\Delta) \sim \Big(\frac{2\pi^2}{3}\Big)^{1/2}\sqrt{\Delta}\,, \tag{5.6}$$

which indeed satisfies the above inequality. In the opposite regime $\Delta/N^2 \gtrsim 1\,(N \gg 1)$, the particle distribution becomes very dilute and hence the above inequality will almost be saturated with some mild corrections. It would be desirable to extend the present analysis to arbitrary finite $N$ and $\Delta$. For a more detailed study of superstar entropy for the single charge case, see [21, 24].

## 6. Discussions

To summarize, with the motivation of constructing a quantum field theory for D-branes in mind, we have first critically reviewed the known fermion representation of 1/2-BPS operators. We have argued the importance of treating all directions of scalar fields on an equal footing. It was shown that this can indeed be achieved by the help of non-renormalization property for the extremal correlators of 1/2-BPS operators. We have then proposed a field theoretic representation of the multi-body system of spherical D3-branes in 1/2-BPS sector. There are two characteristic properties of this extended fermion field theory. One is the necessity of extending Pauli principle to a much more stronger version, coined as 'Dexclusion principle'. This seems to be an inevitable consequence from the factorizable structure of the extremal correlators. This also seems to be consistent with holography, since it has a natural qualitative interpretation on the supergravity side. Another is a rather peculiar nonlocality, intimately related with a superselection rule which can be interpreted as a disguise of the scale symmetry. As such, the nonlocality conforms to the spacetime uncertainty principle, though in the present approximation stringy degrees of freedom do not play roles in any manifest manner.

From the viewpoint of D-brane field theory, what we have done remains yet at a very modest level. It is not clear at this stage whether our construction could really be a starting point towards a more general framework, or merely shows a specialty restricted to the 1/2-BPS sector. If this approach is successful, it would ultimately provide third possible formulation of string theory which could connect open and closed string field theories from a new perspective, providing a new explanation of open-closed string duality: on one hand, it would be a second-quantization of open string-field theory, while on the other hand it would be related to closed-string field theory by the Maldelstam-type duality. See figure 4. It is a great challenge to develop further our formalism to a level where we could envisage such possibilities.

There are many possible directions related to the results of the present work. Let us conclude by listing some of them that have not been mentioned in previous sections.

(1) First of all, our formalism itself looks somewhat artificial yet, for lack of firm principles from which we could hopefully deduce a much tighter structure of the theory uniquely. The complex matrix model has been argued not to be essential for the emergence of the generalized fermion picture, but still has been used as a technical device for dealing with the normal ordering prescription. The method works well for extremal correlators. But it becomes very complicated, once we try to treat non-extremal correlators. There may be better formulation(s) using more appropriate languages for this purpose.

(2) Extension to non-sherical branes and to a satisfactory treatment of general non-extremal correlators is also an important technical problem which must be resolved to make real progress along the present approach. Related to this is the problem of supersymmetric completion of the theory. One of the crucial question would be what is the appropriate generalization of the S-charge symmetry for non-spherical degrees of freedom. Since the S-charge transformation can be interpreted as a sort of 'unitary trick' of the scale transformation, a certain unitary version of $SO(4,2)$ group and its supersymmetric extension might be important.

(3) Extension to non-BPS operators as well as to other BPS operators [25] with smaller supersymmetry: The treatment of full stringy degrees of freedom would perhaps require an extension of the base space into some kind of noncommutative space. It is also an interesting question how the DEP could be extended (or reinterpreted) for non-BPS states which are characterized by anomalous conformal dimensions. We could begin from some appropriate approximations, such as, for instance, the PP-wave limit [15]. The well known relation with spin chains [26] would also be useful. The extension to non-BPS modes will be a prerequisite for the inclusion of anti D-branes in the formalism.

    This also raises the problem of possible nonlinear realization of the maximal $\mathcal{N}=2$ spacetime supersymmetry in 10 dimensions, as we have emphasized in [27] in a more general context. In particular, it is pointed out there that the realization of full 10D $\mathcal{N}=2$ supersymmetry in the presence of both D- and anti-D-branes is closely woven with the open-closed string duality and also with the *s*-*t* channel duality. As stressed above, D-brane field theory is expected to provide a new bridge between open and closed strings. It would be quite worthwhile to revisit the problem from the viewpoint of D-brane field theory.

(4) In connection to the PP-wave limit, we would like to add a further remark. In the LLM classification, the circle at the boundary of the ground state droplet, sitting just at the center ($\rho = 0$) of the $\mathrm{AdS}_5$ spacetime, coincides with the geodesic which is usually used for studying the PP-wave limit. For the purpose of establishing the holographic correspondence of correlators, it is useful [28] to euclideanize the geodesics by performing a Wick rotation for the global time and simultaneously for the angle parametrizing a large circle of $\mathrm{S}^5$. The euclideanized geodesic starts from a point on the AdS boundary and go back to another point of the AdS boundary. This makes possible to directly take the BMN limit of two- and three- point correlation functions through the GKP-Witten relation.[7] After the above Wick rotation, the droplet becomes non-compact, with its boundary being a hyperbola which is the euclidenized geodesic itself and asymptotically reaches the (Euclidean) AdS boundary. The situation is now quite akin to the $c = 1$ matrix model. Indeed, the Wick rotation of the angle parameter effectively changes the sign of harmonic oscillator potential. The scattering amplitude of ripples traveling along the hyperbola should be related to the correlation functions by a definite holographic relation as we have given in [29]. It would be very interesting to connect our D-brane field theory to the 'holographic' string-field theory constructed there by bosonization on the basis of this viewpoint. Of course, we have to extend the usual method [30, 31] of non-relativistic bosonization to include transverse coordinates.

(5) An important question related to this last question is the problem of so-called leg factors, which are usually necessary in making connection between bulk closed strings and matrix models. The recent work [32] seems to be very suggestive in this regard. They have studied a collective-field treatment of matrix operators involving both $Z$ and $Z^\dagger$ in the free-field approximation. From our point of view including all SO(6) directions on an equal footing, these operators should rather be related to the 1/2-BPS operators of the following type

$$\mathrm{Tr}\left(4Z^{J+1}Z^\dagger - \sum_{r=1}^{J} \phi_i Z^r \phi_i Z^{J-r}\right), \tag{6.1}$$

which has conformal dimension $\Delta = J + 2$ and is singlet with respect to SO(4) with the index $i$ running from 1 to 4. In general, we can have similar singlet operators with dimensions $\Delta = J + 2n$ ($n = 1, 2, \ldots$). The authors of [32] have derived a leg factor, which is a generalization of the kernel appearing in the LLM classification and is closely related to the one [33] studied in connection with a deformation of $c = 1$ matrix model in which the deformation parameter may be related to the mass of 2D black hole. In view of mysteries associated with the leg factor in old matrix models, it would be desirable to further study this phenomenon from a new perspective of more general holographic correspondence between bulk and boundary theories.

[7] We warn the reader that this Wick rotation cannot be done as an analytic continuation from the Minkowski formulation, rather should be taken to be a requisite for the possibility of directly taking the BMN limit if we wish to preserve the GKP-Witten relation.

(6) Finally, there are other important questions such as the study of possible M-theory version [6, 9] (or the case of $\mathrm{AdS}_{4,7}\times\ \mathrm{S}^{7,4}$) of the present construction and general (non-conformal) D$p$-branes,[8] formulation of T- and S-dualities, the deformation of backgrounds, and so on. In D-brane field theory, all possible background deformations would *in principle* be represented by fermion bilinear products, so that the notorious problem of treating RR-background fields may be viewed from a new perspective. It has also been suggested [36] in the case of D-instantons that a full-fledged second quantized theory of D-branes may be relevant for approaching the question of background independent formulation.


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## A. A derivation of free fermions for $\Delta = J$ operators

The Hilbert space of the complex 1-matrix quantum mechanics is reducible to a system of $N$ free fermions. Although this has been discussed in the literature [4, 34], we give our own derivation here to set up our notations used in the main text. As discussed in the text, we think that the essence of the emergence of the fermion picture is actually not in the choice of a special two dimensional plane, but in the separation of degrees of freedom into the hermitean 1 matrix model and the kinematical degrees of SO(6). Use of the coherent state representation for the matrix part, however, leads to the completely equivalent formulation as the treatment of complex case given here.

The (euclidean) hamiltonian is

$$H = \mathrm{Tr}\left[ZZ^\dagger + \Pi\Pi^\dagger\right], \tag{A.1}$$

where $Z, Z^\dagger$ and $\Pi, \Pi^\dagger$ are matrix-valued canonical coordinates and momenta satisfying the canonical commuation relations

$$[Z_{nm}, \Pi_{m'n'}] = i\delta_{nn'}\delta_{mm'}\ , \qquad [Z^\dagger_{nm}, \Pi^\dagger_{m'n'}] = i\delta_{nn'}\delta_{mm'} \tag{A.2}$$

with all other commutators vanishing. We use the notation $\dagger$ for matrix conjugartion and $*$ for the quantum-mechanical hermitean conjugation: $Z^\dagger_{mn} = Z^*_{nm}$, $\Pi^\dagger_{mn} = \Pi^*_{nm}$ and also

[8] In connection with this question for the case $p = 0$, the model in [37] where 1/2-BPS states of D3-branes are treated in a framework of DLCQ matrix theories seems to be interesting.



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that in the $Z, Z^\dagger$-diagonal representation

$$\Pi_{mn} = -i\frac{\partial}{\partial Z_{nm}}\,, \qquad \Pi^\dagger_{mn} = -i\frac{\partial}{\partial Z^\dagger_{nm}}\,. \tag{A.3}$$

In this representation, $Z^*_{nm}$ is of course the complex conjugate of $Z_{nm}$. We then define two sets of matrix-valued creation and annihilation operators by

$$A_{nm} \equiv \frac{1}{\sqrt{2}}(Z_{nm} + i\Pi^\dagger_{mn})\,, \qquad A^\dagger_{mn} \equiv \frac{1}{\sqrt{2}}(Z^\dagger_{nm} - i\Pi_{mn})\,, \tag{A.4}$$

$$B_{nm} \equiv \frac{1}{\sqrt{2}}(Z^\dagger_{mn} + i\Pi_{nm})\,, \qquad B^\dagger_{mn} \equiv \frac{1}{\sqrt{2}}(Z_{mn} - i\Pi^\dagger_{nm})\,. \tag{A.5}$$

Only nonvanishing commutators among these oscillators are

$$[A_{nm}, A^\dagger_{m'n'}] = \delta_{nn'}\delta_{mm'} = [B_{nm}, B^\dagger_{m'n'}] \tag{A.6}$$

in terms of which the hamiltonian operator is given as

$$H = \mathrm{Tr}\left[A^\dagger A + B^\dagger B + N\right]. \tag{A.7}$$

Since the 1/2-BPS operators satisfying the condition $\Delta = J$ consist of only the holomorphic coordinate $Z$, we can impose the following condition for allowed states:

$$A_{nm}|\Psi\rangle = \Big(Z_{nm} + \frac{\partial}{\partial Z^*_{nm}}\Big)|\Psi\rangle = 0 \tag{A.8}$$

which in the $Z, Z^\dagger$ diagonal representation leads to

$$\langle Z, Z^\dagger|\Psi\rangle = F[Z]\,\mathrm{e}^{-\mathrm{Tr}(ZZ^\dagger)} \tag{A.9}$$

with $F[Z]$ being an arbitrary $\mathrm{U}(N)$-invariant holomorphic function of the matrix coordinate $Z$. Note that this is nothing but the general form of states in the *coherent-state representation* of the *hermitean* matrix model. The action of the hamiltonian is

$$\begin{aligned}\langle Z, Z^\dagger|H|\Psi\rangle &= \mathrm{Tr}\Big[-\frac{\partial}{\partial Z^\dagger}\frac{\partial}{\partial Z} + Z^\dagger Z\Big]F[Z]\mathrm{e}^{-\mathrm{Tr}(ZZ^\dagger)}\\ &= (HF)[Z]\,\mathrm{e}^{-\mathrm{Tr}(ZZ^\dagger)}\,,\end{aligned} \tag{A.10}$$

where

$$(HF)[Z] = \Big(\sum_{m,n} Z_{nm}\frac{\partial}{Z_{nm}} + N^2\Big)F[Z]\,. \tag{A.11}$$

The holomorphic functions $F[Z]$ are identified one-to-one with the operators $\mathcal{O}^J$. Thus apart from the zero-point contribution $N^2$, the eigenvalue of the hamiltonian is equal to the conformal dimension.

The norm of the holomorphic state is defined by the following internal product

$$\langle\Psi_1|\Psi_2\rangle = \int[dZdZ^\dagger]\langle\Psi_1|Z, Z^\dagger\rangle\langle Z, Z^\dagger|\Psi_2\rangle = \int[dZdZ^\dagger]\mathrm{e}^{-2\,\mathrm{Tr}(ZZ^\dagger)}F_1[Z^\dagger]F_2[Z]\,. \tag{A.12}$$

The conjugate operators $\overline{\mathcal{O}}^J$ are naturally identified with the anti-holomorphic function $F[Z^\dagger]$. It is easy to check that the action of the hamiltonian is self-conjugate in the sense that

$$\begin{aligned}\langle\Psi_1|H|\Psi_2\rangle &= \int [dZdZ^\dagger]\mathrm{e}^{-2\,\mathrm{Tr}(ZZ^\dagger)}F_1[Z^\dagger](HF_2)[Z] \\ &= \int [dZdZ^\dagger]\mathrm{e}^{-2\,\mathrm{Tr}(ZZ^\dagger)}(HF_1)[Z^\dagger]F_2[Z] \end{aligned} \tag{A.13}$$

with

$$(HF)[Z]^\dagger = \Big(\sum_{m,n} Z^*_{nm}\frac{\partial}{Z^*_{nm}} + N^2\Big)F[Z^\dagger]\,. \tag{A.14}$$

Since we are interested in the gauge invariant functions for $F[Z]$, it is natural to diagonalize the matrix coordinates. The symmetry which preserves both the hamiltonian and the general states of the above form is the group of unitary transformations,

$$Z \to UZU^\dagger, \quad Z^\dagger \to UZ^\dagger U^\dagger\,, \qquad UU^\dagger = 1\,.$$

The measure defined above for the internal product of holomorphic states is identical with the old standard form for the ensemble of random complex matrices, known as the Ginibre ensemble [35]. Though it is not possible to diagonalize a generic complex matrix by a unitary transformation, it is always possible to make it to the triangular form

$$Z = U(\Lambda + K)U^\dagger\,, \tag{A.15}$$

where $\Lambda$ is the complex diagonal matrix $\Lambda_{mn} = \delta_{mn}z_n$ with $z_n$'s being the eigenvalues of $Z$ and $K$ is a lower-triangular matrix satisfying $K_{mn} = 0$ for $m \geq n$ (no diagonal elements). For arbitrary (anti-)holomorphic polynomials $F[Z]$ ($F[Z^\dagger]$), this is sufficient for expressing the traces in terms of eigenvalues $z_n$ ($z_n^*$).

$$\mathrm{Tr}(Z^k) = \sum_n z_n^k\,.$$

Up to a numerial factor, the integration measure is then given by

$$[dZdZ^\dagger]\mathrm{e}^{-2\,\mathrm{Tr}(ZZ^\dagger)} = [dU][dKdK^\dagger]\Big(\prod_n dz_n dz_n^*\Big)\Delta[z]\Delta[z^*]\exp\left[-2\sum_n |z_n|^2 - 2\,\mathrm{Tr}(KK^\dagger)\right], \tag{A.16}$$

where $[dU]$ is the invariant measure for the $\mathrm{U}(N)$ group and

$$\Delta[z] = \prod_{n<m}(z_n - z_m)\,. \tag{A.17}$$

For the (anti-)holomorphic trace operators, the degrees of freedom of the lower-triangular part $K$ are decoupled and can be integrated out giving a numerical factor. We warn the reader, however, that if we try to extend our treatment of correlators to non-extremal case, we have to include both holomorphic and anti-holomorphic variables into a single trace. Then, the decoupling of the triangular matrices $K$ is no more valid. Hence, the treatment of normal ordering prescription becomes more complicated as mentioned in the text.

Thus, we are led to define the Hilbert space of multi-particle wave functions of the form

$$\langle z, z^* | \Psi \rangle = f[z]\, \mathrm{e}^{-2\sum_n |z|^2}, \qquad f[z] = \Delta[z] F[z], \tag{A.18}$$

which is sufficient for discussing extremal correlators. The internal product is

$$\langle \Psi_1 | \Psi_2 \rangle = \int \Big( \prod_n dz_n dz_n^* \Big) f_1[z]^* f_2[z]\, \mathrm{e}^{-2\sum_n |z_n|^2}. \tag{A.19}$$

The hamiltonian acting on the holomorphic function $f[z]$ is

$$(hf)[z] = \Delta[z](HF)[Z] = \left( \Delta H \Delta^{-1} f \right)[z]. \tag{A.20}$$

For conjugate states, all the quantities are replaced by their complex conjugates. We find

$$h = \Delta[z] \sum_n z_n \frac{\partial}{\partial z_n} \Delta[z]^{-1} + N^2 = \sum_n z_n \frac{\partial}{\partial z_n} + \frac{1}{2} N(N+1). \tag{A.21}$$

Since $f[z]$ is completely antisymmetric with respect to the exchange of the eigenvalues, the system is equivalent with the system of $N$ free fermions. The set $\{f[z]\}$ of antisymmetric holomorphic functions has a one-to-one correspondence to the set $\{F[Z]\}$ of the products of traces of the complex matrix $Z$. Aside from the constant contribution $N(N+1)/2$, the single-particle hamiltonian is simply

$$h_s = z \frac{\partial}{\partial z} \tag{A.22}$$

whose eigenfunctions are just the monomials $z^n$ with eigenvalue $n$. Therefore the ground-state energy of the $N$ fermions is equal to $\sum_{n=0}^{N-1} 1 = N(N-1)/2$ which gives the correct zero-point energy $N^2$ combined with the constant contribution $N(N+1)/2$. It is well known that this system is formally identical with the one which emerges in describing the (integer) quantum Hall system in two dimensions.

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